WORLD METEOROLOGICAL ORGANIZATION

COMPENDIUM OF METEOROLOGY

For use by
CLASS I AND CLASS II METEOROLOGICAL PERSONNEL

Editor
Aksel Wiin-Nielsen

VOLUME I
PART 1 - DYNAMIC METEOROLOGY

Prepared by
Aksel Wiin-Nielsen

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**FOREWORD**

Following the publication in 1969 by WMO of "Guidelines for the education and training of meteorological personnel" (WMO-No. 258.TP.144) numerous requests were received, as a logical follow up to the publication, for lecture notes based on the syllabi it contained. It was felt that such lecture notes would give Instructors the necessary guidance regarding the teaching standards required by the syllabi and at the same time provide students with valuable material for study and reference appropriate to their level of training.

The implications of such a project were discussed initially by the Executive Committee Panel of Experts on Meteorological Education and Training and subsequently by the twentieth session (1968) of the Executive Committee which agreed to the preparation of Compendia of lecture notes for training Class III and Class IV meteorological personnel. These Compendia were published in English and French during 1970-1972.

This present Compendium originated from a recommendation of the fifth session (1969) of Regional Association I which was greatly concerned at the lack of material available for training Class II meteorological personnel. This recommendation was supported by the Executive Committee Panel of Experts on Meteorological Education and Training and approved for implementation by the twenty-second session (1970) of the Executive Committee.

Early in the initial stages of implementation of this project it was realized that it would be rather difficult to find one author who could write the texts on all parts of the syllabi. For this reason it was decided to avail of the services of a number of scientists each of whom agreed to write the lectures on specific topics of the syllabi. Thus the lectures on particular topics such as dynamic meteorology, physical meteorology, synoptic meteorology, aeronautical meteorology, climatology, hydrology, tropical meteorology, long-range forecasting and interaction of the ocean and the atmosphere have been prepared by individual experts in these fields.
In the course of writing the material a number of authors experienced difficulty in distinguishing between the level of training required by Class II personnel and that by Class I personnel. Indeed the "Guidelines" itself states that:

"Class II meteorological training in the fields of dynamic and synoptic meteorology is essentially the same as that for Class I personnel. In principle therefore the syllabi for these subjects for both classes should be the same. For Class II however topics should be taught with a view to their practical applications."

This general principle applies to other topics also although perhaps to a lesser extent.

This being so, much of the material contained in this Compendium can be used to advantage by both Class I and Class II personnel. Rather than restrict its use to Class II personnel therefore the title originally proposed for this Compendium has been modified slightly to facilitate its use by all students receiving an advanced level education in meteorology.

The complete Compendium will comprise three volumes. This present publication constitutes Volume I and contains lecture material relating to physical meteorology and dynamic meteorology. Volumes II and III will contain relevant material on the remaining topics mentioned above.

In conclusion, I would like to take this opportunity to thank all the authors who have contributed to this Compendium and particularly Mr. B. J. Retallack (Australia) and Professor Aksel Wiin-Nielsen (Denmark) for their contributions to this first Volume.

I would also like to convey to Professor Wiin-Nielsen my sincere appreciation for the excellent manner in which he has carried out his task as Editor of the Compendium.

Finally I would like to thank the Permanent Representatives of all Member countries of the Organization, who, through their support and encouragement, have made this publication possible.

(D. A. Davies)
Secretary-General
PART 1

DYNAMIC METEOROLOGY
INTRODUCTION

Dynamic meteorology is the study of the basic thermodynamic processes which determine the motion of the air and govern the phenomena produced by the motion. It embraces atmospheric motion on all scales.

The material contained in the following sections is based on the syllabus in Dynamic Meteorology for training Class II personnel suggested on pages 98 to 100 of the WMO "Guidelines for the Education and Training of Meteorological Personnel" (WMO-No. 258 TP. 144). This being so, the presentation of the material has been essentially geared to this group of personnel. Also in contrast with many other texts on the subject, detailed derivations of the mathematical formulae are provided. In order to facilitate the understanding of the mathematical derivations, extensive use has been made of vector representation and simple vector algebra; a brief summary of the necessary concepts in this field is given in the Appendix.

In spite of the efforts to keep the text on an elementary level, it has nevertheless turned out that some chapters treat certain aspects of the material in greater depth than would normally be required of the Class II personnel. This is particularly so in Chapter IX dealing with Stationary Circular Vortices and Chapter XII on Numerical Weather Prediction. For this reason I feel that the material can be used to advantage by students at both the Class I and II levels, although the former should use the content as an introductory text only.

The text has relatively few examples from empirical studies and in fact contains no detailed case studies of atmospheric motion on the various scales, the reason being that these aspects are treated by other authors in later parts of the Compendium.

A. Wiin-Nielsen

March 1972
Chapter I

EQUATIONS OF MOTION

1.1 Introduction

The basis of the equations of motion for the atmosphere is Newton's second law which states that the mass of a particle times the acceleration of the particle is equal to the sum of the forces acting on the particle. The purpose of this chapter is to apply Newton's second law to the earth's atmosphere. We consider then an arbitrary, but small volume which for convenience we will assume to have unit mass (1g, for example). Newton's second law states then that the acceleration of the particle is equal to the sum of the forces acting on the particle. We must remember in this formulation to calculate the forces per unit mass.

It must be emphasized that Newton's law is applicable in a so-called inertial system. By this we mean a system which remains fixed relative to the stars. On the other hand, standard meteorological observations are made at points which are fixed on the earth. As an example, we measure the meteorological variables determining the physical state of the atmosphere, at meteorological stations. For each observation we know the position of the station where we have measured temperature, pressure, density and humidity. Similarly, the horizontal wind is measured relative to the earth. It is thus obvious that we must modify Newton's law in such a way that it applies to a system which is fixed relative to the rotating earth. We shall call the velocity and the acceleration in the inertial system the absolute velocity and the absolute acceleration respectively. The corresponding quantities in the system fixed relative to the rotating earth will be called the relative velocity and the relative acceleration. The former quantities will be denoted by a subscript "a", and the latter by subscript "r".

In the general terms stated in the first paragraph we may write Newton's second law as

$$\left( \frac{d\vec{s}}{dt} \right)_a = \sum_{i=1}^{n} \vec{F}_i$$  \hspace{2cm} (1.1.1)

where the left hand side is the absolute acceleration in the absolute system, and the
right hand side is the sum of the forces acting on the particle which has unit mass.

In order to adapt the above equation to the atmosphere we must first study the acceleration and the forces involved. We shall then return to (1.1.1).

1.2  **Kinematics (Velocity and Acceleration)**

Consider an arbitrary system with origin 0 (see Figure 1.1). The vector connecting 0 with the arbitrary point P, i.e. \( \overrightarrow{OP} \), is called the position vector for P.

![Figure 1.1](image)

Let a particle be situated at P at time t. At a later time \( t_1 = t + \Delta t \) the particle will be at \( P_1 \) with position vector \( \overrightarrow{OP_1} \). The mean velocity \( \overrightarrow{v_m} \) during the time interval \( \Delta t \) is defined as

\[
\overrightarrow{v_m} = \overrightarrow{PP_1} = \frac{\overrightarrow{OP_1} - \overrightarrow{OP}}{\Delta t} = \frac{\overrightarrow{R_1} - \overrightarrow{R}}{\Delta t} = \frac{\Delta \overrightarrow{R}}{\Delta t} \tag{1.2.1}
\]

The velocity vector for the point P at time t is defined as the limiting value of \( \overrightarrow{v_m} \) as \( \Delta t \to 0 \), or

\[
\overrightarrow{v} = \lim_{\Delta t \to 0} \frac{\Delta \overrightarrow{R}}{\Delta t} = \frac{d\overrightarrow{R}}{dt} \tag{1.2.2}
\]

The secant \( PP_1 \) will, in general, approach the tangent to the path (i.e. the trajectory of the particle) as \( \Delta t \to 0 \). We may therefore state that the velocity vector is tangent to the trajectory.

The acceleration \( \overrightarrow{a} \) has the same relation to the velocity as the velocity has to the position vector. We may therefore define the acceleration as the change in the velocity vector per unit time, i.e.

\[
\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt} \tag{1.2.3}
\]
Combining (1.2.2) and (1.2.3) it is seen that
\[ \vec{a} = \frac{d^2 \vec{r}}{dt^2} \tag{1.2.4} \]

If we now refer the velocity vector to a specific co-ordinate system with unit vectors \( \vec{i}, \vec{j}, \) and \( \vec{k} \) we have
\[ \vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} + \frac{dz}{dt} \vec{k} \tag{1.2.5} \]

The length of the velocity vector, called the speed, is
\[ v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \frac{1}{2} \sqrt{\vec{v} \cdot \vec{v}} \tag{1.2.6} \]

Since the velocity vector is along the tangent to the trajectory we may write it in the form:
\[ \vec{v} = v\vec{T} \tag{1.2.7} \]

where \( \vec{T} \) is a unit vector along the tangent.

If \( \vec{b} = \vec{b}(t) \) is a vector function of time with the property that
\[ \vec{b} \cdot \vec{b} = \text{const.} \tag{1.2.8} \]

i.e. that the length of \( \vec{b} \) is a constant, we find by differentiation with respect to time that
\[ \vec{b} \cdot \frac{d\vec{b}}{dt} = 0 \tag{1.2.9} \]

which says that \( \vec{b} \) is perpendicular to \( \frac{d\vec{b}}{dt} \), or in other words:

A vector of constant length is normal to its derivative with respect to time.

From (1.2.7) we find that
\[ \vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{v}}{dt} \vec{T} + v \frac{dT}{dt} \tag{1.2.10} \]
The trajectory is shown in Figure 1.2. $P$ is the position at $t = t_0$ while $P_1$ is the position at $t_1 = t_0 + \Delta t$. $\mathbf{T}$ and $\mathbf{T}_1$ are unit vectors along the tangents at $P$ and $P_1$, respectively. The normals at $P$ and $P_1$ intersect at $C$ which is the centre of curvature ($CP$ = the radius of curvature = $R$). The vector $\mathbf{PR} = P_1Q_1 = \mathbf{T}_1$, and thus

$$\mathbf{QR} = \mathbf{PR} - \mathbf{PQ} = \mathbf{T}_1 - \mathbf{T} = \Delta \mathbf{T}$$  \hspace{1cm} (1.2.11)

From the two triangles $PQR$ and $CPP_1$ we find

$$\frac{\Delta \mathbf{T}}{\Delta t \mathbf{T}_1} = \frac{PP_1}{\Delta t \mathbf{CP}} = \frac{\Delta s}{R \Delta t}$$  \hspace{1cm} (1.2.12)

or, in the limit when $\Delta t \to 0$,

$$\left| \frac{d \mathbf{T}}{dt} \right| = \frac{1}{R} \frac{ds}{dt} = \frac{V}{R}$$  \hspace{1cm} (1.2.13)

We note finally that the vector $\Delta \mathbf{T}$ in the limit $\Delta t \to 0$, is directed along the normal $\mathbf{N}$ pointing towards the centre of curvature. This follows from the theorem above since $\mathbf{T}$ is of constant length. We may therefore write

$$\frac{d \mathbf{T}}{dt} = \frac{V}{R} \mathbf{N}$$  \hspace{1cm} (1.2.14)

and it follows from (1.2.14) and (1.2.10) that

$$\mathbf{a} = \frac{dV}{dt} \mathbf{T} + \frac{V^2}{R} \mathbf{N}$$  \hspace{1cm} (1.2.15)
Equation (1.2.15) may therefore be stated as follows:

The acceleration is composed of a tangential acceleration equal in magnitude to the time derivative of the speed, and a normal acceleration, also called the centripetal acceleration, equal in magnitude to the square of the speed divided by the radius of curvature.

1.3 Calculation of Velocity and Position from Acceleration

If the position vector is given as a function of time we find the velocity and the acceleration by differentiation with respect to time. The solution of the opposite problem, i.e. to compute the velocity and the position vector when the acceleration is known is usually more difficult, but it is one of the main problems in dealing with the kinematics of a particle. We shall here restrict ourselves to some examples.

Example 1.3.1: Let
\[ \vec{R} = \vec{C} + \vec{D}t \]
where \( \vec{C} \) and \( \vec{D} \) are vectors independent of time. We find
\[ \vec{v} = \vec{D} \quad \text{and} \quad \vec{a} = 0 \]
Let us, on the other hand, assume that \( \vec{a} = 0 \). We find then through elementary integrations
\[ \vec{v} = \vec{v}_0, \quad \vec{R} = \vec{R}_0 + \vec{v}_0 t, \]
where \( \vec{v}_0 \) and \( \vec{R}_0 \) are constant vectors. Thus, motion along a straight line with a constant speed is the only motion with a vanishing acceleration.

Example 1.3.2: Let
\[ \vec{a} = \vec{a}_0 = \text{const.} \]
We find
\[ \vec{v} = \vec{v}_0 + \vec{a}_0 t, \quad \vec{R} = \vec{R}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a}_0 t^2 \]
Consider as a special case that the motion is in the xz-plane with \( \vec{F}_o = 0, \) \( v_{ox} > 0 \) and \( v_{oz} > 0, \) and \( \vec{a} = \vec{a}_o = -g \hat{k} \)

We find then

\[
\vec{v} = v_{ox} \hat{x} + v_{oz} \hat{k} = v_{ox} \hat{x} + (v_{oz} - gt) \hat{k}
\]

\[
\vec{r} = x \hat{x} + z \hat{k} = v_{ox} t \hat{x} + (v_{oz} t - \frac{1}{2} gt^2) \hat{k}
\]

which are the equations for the motion. The trajectory in the xz-plane is obtained by eliminating time \( t \) in the equations

\[
x = v_{ox} t
\]
\[
z = v_{oz} t - \frac{1}{2} gt^2
\]

We get

\[
z - \frac{1}{2} \frac{v_{oz}^2}{g} = - \frac{1}{2} \frac{g}{v_{ox}^2} \left( x - \frac{v_{oz}}{v_{ox}} \right)^2
\]

The trajectory is consequently a parabola opening downwards.

Example 1.3.3

Let it be assumed that the acceleration is of constant magnitude, i.e.

\( \vec{a} \cdot \vec{a} = a_o^2 = \text{const.} \), and that the acceleration is normal to the velocity, i.e.

\( \vec{a} \cdot \vec{v} = 0 \). In order to find the trajectory we make use of (1.2.15). Scalar multiplication by \( \vec{a} \) gives

\[
\vec{a} \cdot \vec{a} = \frac{d \vec{v}}{dt} \vec{a} \cdot \vec{T} + \frac{v^2}{R} \vec{a} \cdot \vec{N}
\]

Here, \( \vec{a} \cdot \vec{T} = 0 \), because \( \vec{v} \) is directed along \( \vec{T} \) and \( \vec{a} \cdot \vec{v} = 0 \).

We have therefore

\[
a_o^2 = \frac{v^2}{R} a_o
\]

or

\[
R = \frac{v^2}{a_o} = \frac{v^2}{a_o}
\]
The latter relationship holds because of (1.2.9). The trajectory is a circle because the circle is the only curve with constant curvature.

**Example 1.3.4: Harmonic Oscillation.**

Let the point $P$ move in a circle with radius $A$ and constant angular velocity $\omega$.

![Figure 1.3](image.png)

The projection of $P$ on the $x$-axis will move according to the equation

$$x = A \cos(\omega t + \alpha_0)$$

where $\alpha_0$, the phase angle, is determined from the position of $P$ at $t = 0$. $x$ varies in a periodic way with the period

$$T = \frac{2\pi}{\omega}$$

and the frequency

$$\nu = \frac{\omega}{2\pi}$$

We find the velocity and the acceleration to be

$$v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \alpha_0)$$
\[ a_x = \frac{d^2 x}{dt^2} = -\omega^2 A \cos(\omega t + a_0) \]

It can be seen that \( x \) satisfies the homogeneous linear differential equation of second order

\[
\frac{d^2 x}{dt^2} + \omega^2 x = 0
\]

On the other hand, if the above differential equation is given, it can be seen that \( x \) must vary according to the expression

\[ x = A \cos(\omega t + a_0) \]

where \( A \) and \( a_0 \) are the arbitrary constants of integration.

The motion of the projection of \( P \) on the \( y \)-axis is described by

\[ y = A \sin(\omega t + a_0) \]

which also satisfies the same differential equation.

**Example 13.5: Damped oscillation**

Let the motion be described by the equation

\[ x = A e^{-kt} \cos(\omega t + a_0) \]

We can think of this motion as an oscillator with angular velocity \( \omega \), where the amplitude is decreasing exponentially. The velocity is

\[ v_x = -\omega A e^{-kt} \sin(\omega t + a_0) - k A e^{-kt} \cos(\omega t + a_0) \]

and the acceleration can, after reduction, be written in the form

\[ a_x = -(\omega^2 + k^2)x - 2k \frac{dx}{dt} \]

It follows therefore that the motion satisfies the equation

\[
\frac{d^2 x}{dt^2} + 2k \frac{dx}{dt} + (\omega^2 + k^2)x = 0
\]
which is the equation for the damped oscillation.

1.4 Kinematics of Rotating Motion

Let the point P rotate about an axis with an angular velocity \( \Omega \). An example of such a motion is the trajectory of a point, fixed on the earth, as it participates in the rotation of the earth. It is convenient to introduce a so-called rotation vector \( \vec{\Omega} \), defined as a vector pointing along the axis of rotation such that a positive rotation around the axis corresponds to the motion. We have immediately that the speed of the circular motion is equal to

\[
V = \Omega R \sin \theta,
\]

since \( R \sin \theta \) is the radius of the circle. We can now show that the velocity can be written in the form

\[
\vec{V} = \vec{\Omega} \times \vec{R} \tag{1.4.1}
\]

![Figure 1.4](image)

From the definition of the cross-product we find that \( \vec{V} \) has the correct direction and magnitude, proving (1.4.1).

When the speed is constant the acceleration of this motion is purely along the normal (centripetal acceleration) and we find from (1.2.15) that

\[
\vec{a}_c = \frac{V^2}{R} \vec{N} = \Omega^2 R \sin \theta \vec{N} \tag{1.4.2}
\]

By repeated use of the definition of the cross-product it is easily seen that
\[ \vec{a}_o = \vec{\Omega} \times (\vec{\Omega} \times \vec{R}) \quad (1.4.3) \]

1.5 Absolute and Relative Velocity

Let us now consider a somewhat more complex motion, in which a particle \( P \) moves with a velocity \( \vec{v}_r \) relative to a (geometrical) point \( P_0 \). The latter point, \( P_0 \), rotates as described in 1.4 about the axis with constant angular velocity. This motion is described by the velocity field

\[ \vec{v}_o = \vec{\Omega} \times \vec{R} \quad (1.5.1) \]

![Figure 1.5](image)

During a time interval \( \Delta t \), \( P_0 \) will have moved to \( P'_0 \), when \( P'_0 P_0 = \vec{v}_o \Delta t \).

In the same time \( P \) will have moved to \( P' \) where \( P P' = \vec{v}_r \Delta t \). The absolute (or total) displacement is therefore

\[ P_0 P' = (\vec{v}_o + \vec{v}_r) \Delta t \quad (1.5.2) \]

from which by dividing by \( \Delta t \) we get

\[ \vec{v}_a = \vec{v}_r + \vec{v}_o = \vec{v}_r + \vec{\Omega} \times \vec{R} \quad (1.5.3) \]

This important formula says that the velocity \( \vec{v}_a \) in one system is equal to the velocity \( \vec{v}_r \) in another system plus the velocity of the second system relative to the first.
(1.5.3) may also be written in the form

\[
\left( \frac{d\vec{R}}{dt} \right)_a = \left( \frac{d\vec{R}}{dt} \right)_r + \vec{\Omega} \times \vec{R} \tag{1.5.4}
\]

using the definition of velocity. We shall next show that (1.5.4) holds for an arbitrary vector \( \vec{Q} \). We can always write \( \vec{Q} \) in the form

\[
\vec{Q} = \vec{R}_2 - \vec{R}_1 \tag{1.5.5}
\]

where \( \vec{R}_2 \) is the position vector for the endpoint of \( \vec{Q} \) while \( \vec{R}_1 \) is the position vector for the starting point of \( \vec{Q} \). Since \( \vec{R}_2 \) and \( \vec{R}_1 \) are position vectors we have from (1.5.4)

\[
\left( \frac{d\vec{R}_2}{dt} \right)_a = \left( \frac{d\vec{R}_2}{dt} \right)_r + \vec{\Omega} \times \vec{R}_2
\]

\[
\left( \frac{d\vec{R}_1}{dt} \right)_a = \left( \frac{d\vec{R}_1}{dt} \right)_r + \vec{\Omega} \times \vec{R}_1
\]

Subtraction of these two equations leads to

\[
\left( \frac{d\vec{Q}}{dt} \right)_a = \left( \frac{d\vec{Q}}{dt} \right)_r + \vec{\Omega} \times \vec{Q} \tag{1.5.6}
\]

which proves the desired relation.

1.6 Absolute and Relative Acceleration

We can use the newly derived relation (1.5.6) to find the relation between the absolute acceleration of the absolute velocity and the relative acceleration of the relative velocity. The desirability of having such a relation becomes obvious when we realize that Newton's second law applies to the former quantity, while the latter quantity is required for all problems, dealing with observations from the earth's atmosphere.

The desired result follows immediately when it is noted that (1.5.6) is true for an arbitrary vector \( \vec{Q} \). We shall apply (1.5.6) setting \( \vec{Q} = \vec{v}_a \) and recalling (1.5.3). Using the rules of vector algebra, we then have
\[
\frac{d\vec{v}_a}{dt}_a = \left( \frac{d\vec{v}_a}{dt}_r \right)_r + \vec{\Omega} \times \vec{v}_a \\
= \frac{d}{dt} \left[ \vec{v}_r + \vec{\Omega} \times \vec{R} \right]_r + \vec{\Omega} \left[ \vec{v}_r + \vec{\Omega} \times \vec{R} \right]_r \\
= \frac{d\vec{v}_r}{dt}_r + 2\vec{\Omega} \times \vec{v}_r + \vec{\Omega} \times \vec{\Omega} \times \vec{R}
\]  
(1.6.1)

The first term in (1.6.1) is the relative acceleration in the relative system. The velocity measured by meteorological instruments is \( \vec{v}_r \) and in general we are interested in describing the motion relative to the earth. The natural acceleration to use is therefore \( \left( \frac{d\vec{v}_r}{dt} \right)_r \). The additional two terms in (1.6.1) are the accelerations due to the rotation of the earth. The second term \( 2\vec{\Omega} \times \vec{v}_r \) is called the Coriolis acceleration. It is always perpendicular to the relative velocity. The third term in (1.6.1) we recognize as the centripetal acceleration considered in (1.4.3). Whereas the Coriolis acceleration exists only when the relative motion is different from zero, we always have a centripetal acceleration. The magnitude of the latter depends on position only and is \( \Omega^2 R \sin \theta \) (see 1.4.2), where \( \theta \) is co-latitude, or \( \Omega^2 R \cos \varphi \), where \( \varphi \) is latitude. For later reference, it is desirable to estimate the maximum value of the magnitude of the centripetal acceleration. This obviously occurs at the equator (\( \theta = \pi/2, \varphi = 0 \)) and using \( \Omega = 7.29 \times 10^{-5} \text{s}^{-1} \) and \( R = (2/\pi) \times 10^7 \text{m} \) we find \( \Omega^2 R = 0.03 \text{ m s}^{-2} \).

We note already that it is customary to talk about \(-2\vec{\Omega} \times \vec{v}_r\) and \(-\vec{\Omega} \times (\vec{\Omega} \times \vec{R})\) as the Coriolis' force and the centrifugal force, respectively. We shall return to these matters in section (1.11).

1.7 Individual and Local Time Derivative

The acceleration which we have considered in the previous sections has been the acceleration of a particle. We note that it is this acceleration which
enters into Newton's second law (1.1.1). However, in general we do not have the possibility to follow a particular particle in the atmosphere, and for this reason we can not measure $d\mathbf{v}/dt$ directly. From wind observations at a given station we can estimate the change in the wind vector per unit time at the point. This quantity, the change per unit time at a fixed point, is called the local change, or the local acceleration if we are dealing with the wind. The purpose of this section is to determine the relationship between the rate of change of a given quantity following the particle (the individual or the particle time derivative) and the local derivative.

Let $b = b(x, y, z, t)$ be a scalar variable. A particle $P$ is at $t = t_0$ at the geometrical point $(x_0, y_0, z_0)$. At the time $t = t_0 + dt$ the particle is at the point $(x_0 + dx, y_0 + dy, z_0 + dz)$.

We consider now

$$db = b(x_0 + dx, y_0 + dy, z_0 + dz, t_0 + dt) - b(x_0, y_0, z_0, t_0) \quad (1.7.1)$$

which is the change in the property $b$ for the particle in the time interval $dt$.

According to Taylor's expansion (including only terms of the first order) we may write

$$db = \frac{\partial b}{\partial x} dx + \frac{\partial b}{\partial y} dy + \frac{\partial b}{\partial z} dz + \frac{\partial b}{\partial t} dt + \text{h.o.t.} \quad (1.7.2)$$

where h.o.t. means higher order terms. When we divide (1.7.2) by $dt$, in the limit ($dt \to 0$) we obtain

$$\frac{db}{dt} = \frac{\partial b}{\partial t} + u \frac{\partial b}{\partial x} + v \frac{\partial b}{\partial y} + w \frac{\partial b}{\partial z} \quad (1.7.3)$$

where we have already introduced the three velocity components $u = dx/dt$, $v = dy/dt$ and $w = dz/dt$, in an arbitrary system. Defining the ascendent vector as

$$\mathbf{vb} = \frac{\partial b}{\partial x} \hat{\mathbf{i}} + \frac{\partial b}{\partial y} \hat{\mathbf{j}} + \frac{\partial b}{\partial z} \hat{\mathbf{k}} \quad (1.7.4)$$

we can write (1.7.3) in the form
\[ \frac{\delta b}{\delta t} = \frac{\partial b}{\partial t} + \vec{\nabla} \cdot \vec{v} b \quad (1.7.5) \]

which is the desired relation. We can therefore state \(1.7.5\) in the form:

The individual time derivative is equal to the local time derivative plus the convective acceleration \(\vec{\nabla} \cdot \vec{v} b\).

We note from \(1.7.4\) that the ascendant must be evaluated at a fixed time. Let us now consider the value of \(b\) at two different points \((x, y, z)\) and \((x + \delta x, y + \delta y, z + \delta z)\) at the same time. We find that (see Figure 1.7(a))

\[ \delta b = \frac{\partial b}{\partial x} \delta x + \frac{\partial b}{\partial y} \delta y + \frac{\partial b}{\partial z} \delta z \quad (1.7.6) \]

or

\[ \delta b = \vec{v} b \cdot \delta \vec{r} \quad (1.7.7) \]

where \(\delta \vec{r}\) is the vector from the first to the second point.

![Diagram of points and velocity field](image)

(a) \hspace{2cm} (b) \hspace{2cm} (c) 

Figure 1.7

If in particular, we select the two points on the same equiscalar surface (an equiscalar surface is a surface on which \(b\) is constant) (see Figure 1.7(b)) we have \(\delta b = 0\) and \(1.7.7\) becomes

\[ \vec{v} b \cdot \delta \vec{n} = 0. \quad (1.7.8) \]

which means that the ascendant is perpendicular to the equiscalar surface as shown in the Figure. Considering next the case where the two points are selected along the same normal to the equiscalar surface we find (see Figure 1.7(c) from \(1.7.7\)) that

\[ \delta b = \vec{v} b \cdot \delta \vec{n} = |\vec{v} b| \cdot \delta n \quad (1.7.9) \]

where \(|\vec{v} b|\) is the magnitude of the ascendant and \(\delta n\) is the normal distance between the two equiscalar surfaces. \(1.7.9\) written in the form
(1.7.10)

is often used in practice to calculate the magnitude of the ascendent of a scalar quantity from the analysis in two or three dimensions.

Having derived the properties of the ascendent, we notice that the convective acceleration is due to the fact that the flow, in general, passes through the equiscalar surface. In the special case, where the flow is in the equiscalar surface, we find \( \mathbf{v} \cdot \nabla b = 0 \) (i.e. a vanishing convective acceleration) in which case \( \frac{dB}{dt} = \frac{\partial b}{\partial t} \). We further note that the field \( b \) is termed permanent or stationary if \( \frac{\partial b}{\partial t} = 0 \), and conservative, if \( \frac{db}{dt} = 0 \).

**NOTE:** The material covered in sections 1.2 to 1.7, inclusive, is related to kinematics. The major results of these sections are (1.6.1) giving the conversion from the absolute to the relative system, and (1.7.5) giving the conversion from the individual to the local derivative. These sections therefore deal with adapting the acceleration in Newton's second law to the earth's atmosphere. In the following sections we shall consider the forces which enter into (1.1.1).

1.8 **The Force of Gravitation**

In general the most important force in geophysics is the attraction due to the masses. The form of this law for two particles of masses \( m_1 \) and \( m_2 \) can be stated as follows:

The attraction of \( m_1 \) on \( m_2 \) is directed from \( m_2 \) towards \( m_1 \) and is proportional to \( m_1 \) and \( m_2 \) and inversely proportional to the square of the distance between them (see Figure 1.8)

![Figure 1.8](image)

We can write this statement in the form

\[
\mathbf{g}_a = -G \frac{m_1}{r^2} \mathbf{r} \tag{1.8.1}
\]

where \( \mathbf{g}_a \) is the force of gravitation, \( m_2 = 1 \), and \( \mathbf{r} \) is the vector from \( m_1 \) to \( m_2 \). \( G \) is the universal gravitational constant. Note, that \( \mathbf{g}_a \) denotes gravitation only
when \( m_2 = 1 \). \( G = 6.66 \times 10^{11} \text{ newton m}^2 \text{ kg}^{-2} \).

It is now necessary to find the resultant gravitation when we add up all the contributions from all the mass particles of the earth. In the first approximation we may consider the earth as a sphere in which the density is a function of the distance from the centre. For such a body it is possible to show that the collected effects of all particles is equal to the gravitational force which we get if we consider the earth as a mass point where the total mass is situated at the centre of gravity. The total gravitational attraction of the complete earth on a unit particle in the atmosphere may therefore be written:

\[
\vec{g_a} = -G \frac{M}{r^2} \vec{r}
\]

where \( M = 5.988 \times 10^{24} \text{ kg} \) is the total mass of the earth and \( \vec{r} \) is the vector from the centre of the earth to the particle in the atmosphere. We note that \( \vec{g_a} \) is directed towards the centre of the earth, that it depends only on distance from the centre of gravity, and that \( \vec{g_a} \) is therefore constant on concentric spheres. We may naturally also describe \( \vec{g_a} \) by the gradient of a scalar potential function \( \phi \) such that

\[
\vec{g_a} = -\nabla \phi_a, \quad \phi_a = -GM \frac{1}{r}, \text{ const.}
\]

1.9 The Pressure Force

The pressure force is due to the variation of the atmospheric pressure from point to point. We recall that pressure is defined as the force per unit area acting by the surroundings on a given surface element. If the area of a given element is \( d\sigma \) the pressure force is \( p \ d\sigma \). It can be shown experimentally that the pressure is the same in all directions at a given point in the fluid. We can see this if we consider the tetrahedron PABC (Figure 1.9). Let ABC have the area \( d\sigma \) and let \( \vec{n} \) be a normal unit vector, directed outwards from PABC. The following pressure forces will act on PABC (See Figure 1.9):

- On PBC: \( \vec{F_p} \text{ area (PBC)} = \vec{F_p} \ d\sigma \cos (n,x) \)
- On PAC: \( \vec{F_p} \text{ area (PAC)} = \vec{F_p} \ d\sigma \cos (n,y) \)
- On PAB: \( \vec{F_p} \text{ area (PAB)} = \vec{F_p} \ d\sigma \cos (n,z) \)
- On ABC: \( \vec{F_p} \text{ area (ABC)} = -\vec{F_p} \ d\sigma \)
The equation of motion for PABC is

\[ m(\vec{a} - \vec{F}_E) = \vec{F} \]  

(1.9.1)

where \( m \) is the mass, \( \vec{a} \) the acceleration, \( \vec{F}_E \) the total external force, and \( \vec{F} \) the total pressure force. Let us now consider what happens if we let the volume of PABC tend to zero. If the acceleration and the external forces are finite, (and we will assume that this is the case) the left side tends to zero as \( L^3 \) where \( L \) is the linear dimension of PABC. Also, the right-hand side of (1.9.1) must tend to zero as \( L^3 \). However, the pressure components tend to zero as \( L^2 \) since the pressure force is proportional to area. Therefore the total pressure force must tend to zero as \( L \to 0 \). Consequently:

\[ \vec{n}_x \cos(n,x) + \vec{n}_y \cos(n,y) + \vec{n}_z \cos(n,z) = \hat{n}_\sigma \]  

(1.9.2)

but

\[ \vec{n} = \hat{x}_x \cos(n,x) + \hat{y}_y \cos(n,y) + \hat{z}_z \cos(n,z) \]  

(1.9.3)

Substituting from (1.9.3) in (1.9.2) we find that

\[ p_x = p_y = p_z = p_\sigma = p \]  

(1.9.4)
which says that the pressure at a given point and at a given time is independent of direction.

![Diagram](image)

We seek now the net pressure force per unit mass. To this end we consider the box in Figure 1.10. The centre of the box, point O, has the coordinates \(x, y, z\) and the dimensions of the box are \(\delta x\), \(\delta y\) and \(\delta z\). The net force in the x-direction is

\[
P_1 \delta y \delta z - P_2 \delta y \delta z = \left[ \left( p_o - \frac{1}{2} \frac{\partial p}{\partial x} \delta x \right) - \left( p_o + \frac{1}{2} \frac{\partial p}{\partial x} \delta x \right) \right] \delta y \delta z
\]

\[
= - \frac{\partial p}{\partial x} \delta V, \text{ where } \delta V = \delta x \delta y \delta z
\]

\[\text{(1.9.5)}\]

The mass of the air in the box is \(\rho \delta V\), where \(\rho\) is the density. The net pressure force per unit mass in the x-direction is therefore \(-1/\rho \left( \partial p / \partial x \right)\). Considering the components of the pressure force in the y- and z-directions in a similar way we find that the net pressure force is

\[
- \frac{1}{\rho} \left( \frac{\partial p}{\partial x} I + \frac{\partial p}{\partial y} J + \frac{\partial p}{\partial z} K \right) = - \frac{1}{\rho} \nabla p
\]

\[\text{(1.9.6)}\]

Since the ascendent \(\nabla p\) is normal to the isobaric surfaces and directed from lower to higher pressure, we find that the pressure force in (1.9.6) is also normal to the isobaric surfaces, but directed from higher to lower pressure.

1.10 Frictional Forces

Later, in Chapter XIII, we shall consider the major modifications of the atmospheric flow caused by the frictional forces. We shall be interested then in the specific form of the frictional forces created by the fact that the atmosphere is a turbulent medium. However, in this chapter we shall consider the main nature of the frictional force. It will suffice to consider a relatively simple example initially.
Let us assume that the motion is horizontal \( w = 0 \) and that the spatial variation is in the \( z \)-direction only, i.e.

\[
\mathbf{v} = u(z) \mathbf{i} + v(z) \mathbf{j}
\]  

(1.10.1)

Figure 1.11 illustrates the even simpler case where \( v = 0 \). Because of the internal properties of a fluid or a gas we can expect that the fluid at one height will try to accelerate the fluid below. On the other hand, the fluid below will try to slow down the fluid above if the latter moves faster than the former.

We shall now make the basic assumption that the accelerating force is proportional to \( \mathbf{v}_2 - \mathbf{v}_1 \) where \( \mathbf{v}_2 \) is the velocity just above and \( \mathbf{v}_1 \) the velocity just below a given level. This level is indicated by a dashed line in Figure 1.11. Under the assumption that \( \mathbf{v}(z) \) has a continuous variation we may write

\[
\mathbf{v}_2 - \mathbf{v}_1 = \frac{d\mathbf{v}}{dz} dz
\]  

(1.10.2)

and, according to the basic assumption, we may write the frictional force per unit area (also called the stress) in the form

\[
\tau = \mu \frac{d\mathbf{v}}{dz}
\]  

(1.10.3)

where \( \mu = \mu(z) \) is the coefficient of viscosity. The stress acts along a surface as opposed to the pressure which acts normal to the surface.

Assuming the form (1.10.3) we shall now find the frictional force per unit mass. We consider then a box as given in Figure 1.12.
The force on the top side is \( \tau(z + dz) \) and on the bottom is \( -\tau(z) \).

The resulting force is

\[
\tau(z + dz) - \tau(z) = \frac{d\tau}{dz} dz
\]

(1.10.4)

The volume is \( 1 \cdot dz \) and the mass is \( \rho dz \) of the air in the box. The force per unit mass is therefore

\[
\frac{1}{\rho} \frac{d\tau}{dz} = \frac{1}{\rho} \frac{d}{dz} \left[ \mu \frac{d\mathbf{v}}{dz} \right]
\]

(1.10.5)

In the simple example treated above we have considered only the horizontal components of the stress, and these were considered only in the case where \( \mathbf{v} = \mathbf{v}_h(z) \). It is obvious therefore that the considerations become much more complex when the wind also has a vertical component \( w \), and when \( \mathbf{v} \) varies with all the space co-ordinates \( x, y \) and \( z \). A closer examination shows that the general case can be handled using the so-called stress tensor, but this subject is too advanced for the present treatment. Suffice to say that the component given in (1.10.5) is normally the largest component, since the main variation of the horizontal wind is in the vertical direction.

The dimension of \( \mu \) can be obtained from (1.10.3) where \( \tau \) has the dimension of force per unit area and \( \frac{d\mathbf{v}}{dz} \) has the dimension of time\(^{-1}\). For air at 0°C the value of \( \mu = 1.7 \times 10^{-10} \text{ N m}^{-2} \text{ s}^{-1} \). This number will be important when we consider turbulent flow later.
1.11 The Equations of Motion

The kinematics in sections 1.2 to 1.7 have now been extended in sections 1.8 to 1.10 to a consideration of the major forces in the atmosphere. We may now return to equation (1.11.1) and collect all the partial results in the equations of motion.

We find

\[
\frac{d\vec{v}}{dt} + 2\vec{\Omega} \times \vec{v} + \vec{R} \times (\vec{\Omega} \times \vec{R}) = -\frac{1}{\rho} \nabla \rho \cdot \vec{r} + \frac{1}{\rho} \frac{\partial \vec{r}}{\partial z} \quad (1.11.1)
\]

where we have dropped the subscript "r" on the relative velocity. It is customary to move the Coriolis and the centripetal acceleration to the right hand side of (1.11.1) and rename them the Coriolis and the centrifugal forces, respectively. (1.11.1) takes the form

\[
\frac{d\vec{v}}{dt} = -\frac{1}{\rho} \nabla \rho \cdot \vec{r} - 2\vec{\Omega} \times \vec{v} + \vec{g}_a - \vec{R} \times (\vec{\Omega} \times \vec{R}) + \frac{1}{\rho} \frac{\partial \vec{r}}{\partial z} \quad (1.11.2)
\]

We note in (1.11.2) that the gravitation and the centrifugal force are the only two forces which depend solely on position. It is therefore usual and natural to combine these forces into one force, called gravity, i.e.

\[
\vec{g} = \vec{g}_a - \vec{R} \times (\vec{\Omega} \times \vec{R}) \quad (1.11.3)
\]

Figure 1.13 shows the directions of \(\vec{g}_a\), \(-\vec{\Omega} \times (\vec{\Omega} \times \vec{R})\) and \(\vec{g}\). We can easily obtain an idea of the relative and absolute magnitudes of the vectors in (1.11.3). Using (1.8.2) for \(\vec{g}_a\) with \(G = 6.66 \times 10^{-11}\) N m\(^2\) kg\(^{-2}\), \(M = 5.983 \times 10^{24}\) kg and \(r = 0.6366 \times 10^{-7}\) m we find \(g_a = 9.84\) m s\(^{-2}\) which is the gravitational attraction.
per unit mass at mean sea-level. On the other hand, using \( \Omega = 7.3 \times 10^{-5} \text{s}^{-1} \) and the same value of \( r \) we find that \( \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \)\(^\dagger\) = 0.05 m s\(^{-2}\) at the equator. The total variation of \( \mathbf{g} \) from the pole to the equator along a spherical surface is therefore quite small, and in many problems we may consider the magnitude of \( \mathbf{g} \) as a constant.

It must also be pointed out that the earth is not a perfect sphere, but rather a body which has been deformed in such a way that \( \mathbf{g} \) is everywhere normal to the surface, which therefore is an equiscalar surface for the geopotential. We may think of this as a result of the centrifugal force which has produced a slight bulge at the equator and a slight flattening at the poles. The earth is therefore rather an oblate spheroid than a sphere. However, the earth may be assumed spherical in almost all meteorological problems.

Introducing (1.11.3) in (1.11.2) we get

\[
\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla \rho - 2 \mathbf{\Omega} \times \mathbf{v} + \mathbf{g} + \frac{1}{\rho} \frac{\partial \mathbf{r}}{\partial z} \quad (1.11.4)
\]

In most practical problems we must replace (1.11.4) by three component equations in a suitable co-ordinate system. We obtain therefore three scalar equations which describe the time developments of \( u, v \) and \( w \), the three components of velocity. However, it is obvious that (1.11.4) contains five dependent variables, i.e. \( u, v, w, p \) and \( \rho \). \( \mathbf{g} \) and \( \mathbf{\Omega} \) are known, and \( \mathbf{r} \) is expressed in terms of known functions such as frictional coefficients and \( \mathbf{v} \). It follows therefore that additional equations are needed in order to close the system, i.e. to have as many equations as there are dependent variables in the system. The additional equations will be derived in Chapter II.

It is obviously the most natural procedure to write the component equations from (1.11.4) in a system using spherical co-ordinates. However, due to the mathematical difficulties created by the fact that this co-ordinate system is non-cartesian, we shall first consider another case. For many purposes it is sufficient to consider a local cartesian co-ordinate system, where the \( x \)-axis points east, and the \( y \)-axis north and the \( z \)-axis is directed positively upwards opposite to the local direction of gravity. We may therefore note that the \( xy \)-plane is normal to the gravity vector. Figure 1.14 shows the definitions of the co-ordinate system.
The object is now to find the $x$, $y$ and $z$ components of each term in (1.11.4). We have

$$ \frac{d\mathbf{v}}{dt} = \frac{du}{dt} \mathbf{i} + \frac{dv}{dt} \mathbf{j} + \frac{dw}{dt} \mathbf{k} $$

$$ - \frac{1}{p} \mathbf{v} \times \mathbf{p} = - \frac{1}{p} \frac{dp}{dx} \mathbf{i} - \frac{1}{p} \frac{dp}{dy} \mathbf{j} - \frac{1}{p} \frac{dp}{dz} \mathbf{k} $$

$$ \mathbf{g} = - \mathbf{g} \mathbf{k} $$

$$ \frac{1}{p} \mathbf{F} = \frac{1}{p} \mathbf{F}_x \mathbf{i} + \frac{1}{p} \mathbf{F}_y \mathbf{j} + \frac{1}{p} \mathbf{F}_z \mathbf{k} $$

In order to find the components of $-2\Omega \times \mathbf{v}$ we note from Figure 1.14 that $\Omega_x = 0$, $\Omega_y = \Omega \cos \varphi$ and $\Omega_z = \Omega \sin \varphi$. We find then

$$ -2\Omega \times \mathbf{v} = -2 \begin{vmatrix} i & j & k \\ 0 & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix} $$

$$ = -2(\Omega_y v - \Omega_z w) i + 2(-\Omega_u) j - 2(-\Omega_v) k $$

$$ = (2\Omega \sin \varphi - 2\Omega \cos \varphi) i - 2\Omega \sin \varphi j + 2\Omega \cos \varphi k $$

With these intermediate results we may write (1.11.4) as the system

$$ \frac{du}{dt} = - \frac{1}{p} \frac{dp}{dx} + f v - e w + \frac{1}{p} F_x $$

$$ \frac{dv}{dt} = - \frac{1}{p} \frac{dp}{dy} - f u + \frac{1}{p} F_y $$

$$ \frac{dw}{dt} = - \frac{1}{p} \frac{dp}{dz} + e u - g + \frac{1}{p} F_z $$

(1.11.5)
where we have used the notation \( f = 2 \Omega \sin \varphi \) and \( e = 2 \Omega \cos \varphi \). (1.11.5) applies in a local cartesian co-ordinate system. The origin can be selected arbitrarily, but once determined, it is fixed. It can be seen from Figure 1.14 that the xy-plane will deviate from the earth's surface when we move away from the origin. The system is therefore most suitable for problems where the horizontal scale is not too large. When we consider large-scale problems such as planetary waves where the horizontal dimension is comparable with the radius of the earth, it is advisable to use spherical co-ordinates.

It is useful to make an estimate of the order of magnitude of the terms in (1.11.5) and to investigate whether any of them can be neglected. For example, in the third equation we may neglect \( eu \) compared to \( g \) since \( g \approx 10 \, \text{m s}^{-2} \), while

\[
|eu|/|ul|_{\text{max}} = 14.58 \times 10^{-5} \times 10^2 \approx 1 \times 10^{-2}
\]

where \( |ul|_{\text{max}} = 100 \, \text{m s}^{-1} \). It is thus seen that even the excessive speed of 100 m s\(^{-1}\) will make \( eu \) only three orders of magnitude smaller than \( g \). Large values of \( w \) and of \( dw/dt \) occur in thunderstorms and other small scale systems. On the other hand, if we restrict our considerations to systems of large scale we can normally neglect the acceleration \( dw/dt \). The reason is that \( |w| \) does not normally exceed 0.10 m s\(^{-1}\) and the typical time scale of the large scale systems is large. To illustrate this point, let us assume that the vertical velocity of a particle went from -0.10 m s\(^{-1}\) to +0.10 m s\(^{-1}\) in 2 sec, i.e. \( dw/dt = 0.10 \, \text{m s}^{-2} \). Even in this very extreme case, \( dw/dt \) would be two orders of magnitude smaller than \( g \). When it is realized that the time from minimum to maximum is normally several hours it is understandable that \( dw/dt \) can be neglected relative to \( g \). Furthermore, it can be shown that the frictional force in the vertical direction is small compared to \( g \) in large-scale motion. Thus, for such type of motion the third equation in (1.11.5) reduces to

\[
0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad (1.11.6)
\]

expressing a balance between gravity acting downwards and the vertical component of the pressure force, acting upwards. Later we shall consider the use of the hydrostatic equation.

If the term \( eu \) is neglected in the third equation of (1.11.5) it follows that the term \( ew \) must be neglected in the first equation of the same system. The reason is that we must maintain energetical consistency in the equations. In order to see this we shall denote the kinetic energy per unit mass by \( k \), i.e.
\[ k = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} (u^2 + v^2 + z^2) \quad (1.11.7) \]

From this definition of \( k \) we get
\[
\frac{dk}{dt} = u \frac{du}{dt} + v \frac{dv}{dt} + w \frac{dw}{dt} \quad (1.11.8)
\]

The equation for \( \frac{dk}{dt} \) is obtained from (1.11.5) by a multiplication by \( u \) in the first equation, \( v \) in the second, \( w \) in the third and subsequent addition. We get
\[
\frac{dk}{dt} = -\frac{1}{\rho} \mathbf{v} \cdot \nabla p - gw + \frac{1}{\rho} \mathbf{v} \cdot \mathbf{F} \quad (1.11.9)
\]

The terms on the right-hand side of (1.11.9) represent the work per unit time of the pressure force, gravity and the frictional force, respectively. It is thus seen that the Coriolis force makes no contribution to the rate of change of the kinetic energy because the force can do no work, since it is perpendicular to the velocity.

Now, if we were to neglect \( eu \) in the third equation, but keep \( ev \) in the first equation, we would get a fictitious source of kinetic energy.

The neglect of \( ew \) in the first equation can naturally also be justified by comparing it to the term \( fv \). We get
\[
\frac{|ew|}{|fv|} = \frac{\cos \varphi}{\sin \varphi} \frac{|w|}{|v|} = \cotan \varphi \frac{|w|}{|v|} \quad (1.11.10)
\]

We see that \(|ew| < |fv|\) in middle and high latitudes as long as \(|w| < |v|\). The latter relationship is normally satisfied since \(|w| \sim 0.10 \text{ m/s}^2\) while \(|v| \sim 10 \text{ m/s}^2\).

Adopting all of the approximations mentioned in this section we may write (1.11.5) in the form
\[
\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv + \frac{1}{\rho} F_x \quad (1.11.11)
\]
\[
\frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu + \frac{1}{\rho} F_y
\]
\[0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g\]
The latter system is quite often used in analysis of dynamical problems. We shall make use of it in most parts of the following text.

1.12  **Spherical Co-ordinates**

The cartesian local co-ordinate system introduced in this chapter is sufficient for many purposes. However, the earth is an approximate sphere, and it is important to use the natural co-ordinate for the earth in applications when the scale is so large that the local cartesian co-ordinate system gives too great a distortion.

![Diagram of spherical coordinates](image)

Figure 1.15

The three co-ordinates are longitude, \( \lambda \), normally measured positive eastward from Greenwich, where \( \lambda = 0 \), latitude \( \phi \), which is measured from zero at the equator to + \( \pi/2 \) at the North pole and - \( \pi/2 \) at the South pole, and distance \( r \) from the centre of the sphere (see Figure 1.15). Since the atmosphere is very shallow compared with the radius of the earth we may replace \( r \) by \( a \), the radius of the earth, when \( r \) is undifferentiated.

We introduce three unit co-ordinate vectors \( \hat{i} \), \( \hat{j} \) and \( \hat{k} \), where \( \hat{i} \) points towards east along a latitude circle, \( \hat{j} \) northwards along a meridian and \( \hat{k} \) positive upwards. The main point is now that as a particle moves it will arrive in a new position, where \( \hat{i} \), \( \hat{j} \) and \( \hat{k} \) will have new directions, i.e. \( \hat{i} \), \( \hat{j} \) and \( \hat{k} \) are functions of position. We notice furthermore that a distance in the x-direction is:
dx = r \cos \varphi \, d\lambda \propto a \cos \varphi \, d\lambda, \text{ in the } y\text{-direction: } dy = rd\varphi = a \, d\varphi \text{ and in the } z\text{-direction: } dz = dr.

We have as before

\[ \vec{v} = u \vec{i} + v \vec{j} + w \vec{k} \quad (1.12.1) \]

but while \( \vec{i}, \vec{j} \) and \( \vec{k} \) are constants in the local cartesian co-ordinate system, they will vary with time in the spherical co-ordinate system. We get therefore

\[ \frac{d\vec{v}}{dt} = \frac{du}{dt} \vec{i} + \frac{dv}{dt} \vec{j} + \frac{dw}{dt} \vec{k} + u \frac{d\vec{i}}{dt} + \frac{d\vec{j}}{dt} + w \frac{d\vec{k}}{dt} \quad (1.12.2) \]

and the difficult part is now to evaluate \( \frac{d\vec{i}}{dt}, \frac{d\vec{j}}{dt} \) and \( \frac{d\vec{k}}{dt} \).

We have first

\[ \frac{d\vec{i}}{dt} = u \frac{\partial \vec{i}}{\partial x} + v \frac{\partial \vec{i}}{\partial y} + w \frac{\partial \vec{i}}{\partial z} \quad (1.12.3) \]

where we have left out \( \frac{\partial \vec{i}}{\partial t} = 0 \). A consideration of Figure 1.16(a) shows that \( \frac{\partial \vec{i}}{\partial y} = \frac{\partial \vec{i}}{\partial z} = 0 \). One can see this by imagining small displacements in the \( y \)- and \( z \)-directions, respectively, and

---

**Figure 1.16(a)**

**Figure 1.16(b)**

notice that \( \vec{i} \) does not change its direction, because it will point perpendicularly and into the plane of the paper at points A, B and C. However, this will not be the case when we make a small displacement in the \( x \)-direction as can be seen from
Figure 1.16(b) where we see the plane of the latitude circle from the axis of rotation. We find from the figure

\[
\left| \frac{\Delta \mathbf{r}}{\partial x} \right| = \lim_{\Delta x \to 0} \frac{\Delta \mathbf{r}}{\Delta x} = \lim_{\Delta \lambda \to 0} \frac{\Delta \lambda}{R \Delta \lambda} = \frac{1}{R} \tag{1.12.4}
\]

and note furthermore that \( \lim(\Delta \mathbf{r}) \) is directed opposite to the vector \( \mathbf{R} \). We find then from Figure 1.16(b) that

\[
\frac{\partial \mathbf{r}}{\partial x} = \frac{\sin \varphi}{R} \mathbf{j} - \frac{\cos \varphi}{R} \mathbf{k} \tag{1.12.5}
\]

from which it follows (\( R = a \cos \varphi \))

\[
\frac{d\mathbf{r}}{dt} = u \frac{\partial \mathbf{r}}{\partial x} = \frac{\tan \varphi}{a} \mathbf{j} - \frac{u}{a} \mathbf{k} \tag{1.12.6}
\]

and finally

\[
u \frac{d\mathbf{r}}{dt} = \frac{u^2}{a} \tan \varphi \mathbf{j} - \frac{u^2}{a} \mathbf{k} \tag{1.12.7}
\]

We next turn to

\[
\frac{d\mathbf{r}}{dt} = u \frac{\partial \mathbf{r}}{\partial x} + v \frac{\partial \mathbf{r}}{\partial y} + w \frac{\partial \mathbf{r}}{\partial z} \tag{1.12.8}
\]

A consideration of Figure 1.16(a) shows that \( \frac{\partial \mathbf{j}}{\partial z} = 0 \), but \( \frac{\partial \mathbf{j}}{\partial x} \neq 0 \) and \( \frac{\partial \mathbf{j}}{\partial y} \neq 0 \). We find from Figure 1.17 that

![Figure 1.17](image1.png)

![Figure 1.18](image2.png)
\[
\left| \frac{d\mathbf{j}}{dx} \right| = \lim_{\Delta x \to 0} \frac{\Delta j}{\Delta x} = \frac{\Delta j}{(AO)\Delta \beta} = \frac{1}{AO} \tag{1.12.9}
\]

From the triangle CAO we find

\[AO = \frac{a}{\tan \psi} \tag{1.12.10}\]

and

\[
\left| \frac{d\mathbf{j}}{dx} \right| = \frac{\tan \psi}{a} \tag{1.12.11}
\]

and it follows that

\[
\frac{d\mathbf{j}}{dx} = -\frac{\tan \psi}{a} \frac{\mathbf{i}}{\mathbf{a}} \tag{1.12.12}
\]

We find next from Figure 1.18 that

\[
\left| \frac{d\mathbf{j}}{dy} \right| = \lim_{\Delta y \to 0} \frac{\Delta j}{\Delta y} = \frac{\Delta j}{a\Delta \phi} = \frac{1}{a} \tag{1.12.13}
\]

and

\[
\frac{d\mathbf{j}}{dy} = -\frac{1}{a} \frac{\mathbf{k}}{\mathbf{k}} \tag{1.12.14}
\]

Collecting the results from (1.12.12) and (1.12.14) we find that

\[
\frac{d\mathbf{j}}{dt} = -\frac{u}{a} \tan \phi \frac{\mathbf{i}}{\mathbf{i}} - \frac{v}{a} \frac{\mathbf{k}}{\mathbf{k}} \tag{1.12.15}
\]

and

\[
v \frac{d\mathbf{j}}{dt} = -\frac{uv}{a} \tan \phi \frac{\mathbf{i}}{\mathbf{i}} - \frac{v^2}{a} \frac{\mathbf{k}}{\mathbf{k}} \tag{1.12.16}
\]

We turn finally to

\[
\frac{dk}{dt} = u \frac{dk}{dx} + v \frac{dk}{dy} + w \frac{dk}{dz} \tag{1.12.17}
\]

where we note that \(dk/dz = 0\).

We find from Figure 1.19 that

\[
\left| \frac{d\mathbf{k}}{dx} \right| = \lim_{\Delta y \to 0} \frac{\Delta k}{a\Delta y} = \frac{1}{a} \tag{1.12.18}
\]
and
\[
\frac{\partial \mathbf{k}}{\partial x} = \frac{1}{a} \mathbf{i}
\]  
(1.12.19)

A consideration of Figure 1.20 results in the relation
\[
\left| \frac{\partial k}{\partial y} \right| = \lim_{\Delta y \to 0} \frac{\Delta k}{\Delta y} = \frac{\Delta \varphi}{a \Delta y}
\]  
(1.12.20)

and finally
\[
\frac{\partial \mathbf{k}}{\partial y} = \frac{1}{a} \mathbf{j}
\]  
(1.12.21)

We get therefore
\[
\frac{d\mathbf{k}}{dt} = \frac{u}{a} \mathbf{i} + \frac{v}{a} \mathbf{j}
\]  
(1.12.22)

and finally
\[
\mathbf{w} \frac{d\mathbf{k}}{dt} = \frac{uw}{a} \mathbf{i} + \frac{vw}{a} \mathbf{j}
\]  
(1.12.23)

Figure 1.19  

Figure 1.20

Inserting from (1.12.7), (1.12.16) and (1.12.23) into (1.12.2) we find by gathering separately the contributions in the $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ directions that
\[ \frac{d\vec{v}}{dt} = \left( \frac{du}{dt} - \frac{uv}{a} \tan \varphi + \frac{uw}{a} \right) \hat{r} + \left( \frac{dv}{dt} + \frac{u^2}{a} \tan \varphi + \frac{vw}{a} \right) \hat{j} + \left( \frac{dw}{dt} - \frac{u^2}{a} - \frac{v^2}{a} \right) \hat{k} \]  

(1.12.24)

We may now write the equations of motion in spherical co-ordinates corresponding to the system (1.11.5) before approximations are made:

\[ \frac{du}{dt} = - \frac{1}{\rho a \cos \varphi} \frac{\partial p}{\partial \lambda} + f v - e w + \frac{uv}{a} \tan \varphi - \frac{uw}{a} + \frac{1}{\rho} F_x \]  

(1.12.25a)

\[ \frac{dv}{dt} = - \frac{1}{\rho} \frac{\partial p}{\partial \varphi} - f u - \frac{u^2}{a} \tan \varphi - \frac{vw}{a} + \frac{1}{\rho} F_y \]  

(1.12.25b)

\[ \frac{dw}{dt} = - \frac{1}{\rho} \frac{\partial p}{\partial z} - g + e u + \frac{u^2}{a} + \frac{v^2}{a} + \frac{1}{\rho} F_z \]  

(1.12.25c)

We might mention furthermore that the individual derivative may be written

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \varphi} + \frac{w}{a} \frac{\partial}{\partial z} \]  

(1.12.26)

It is of importance to note that the kinetic energy equation (1.11.9) still holds, because all the new terms in (1.12.25) cancel two and two (see the separate underlining in the equations). In connexion with the approximations leading to the system (1.11.11) we note that \( u^2/a \) and \( v^2/a \) are of the order of magnitude \( 10^{-5} \text{ m s}^{-2} \) if \( u \approx v \approx 10 \text{ m s}^{-1} \). They are therefore very small compared with \( g \approx 10 \text{ m s}^{-2} \), and the hydrostatic equation is still very well satisfied. If the third equation in (1.12.25) is reduced to the hydrostatic equation we must, for energetic consistency, neglect the double and triple underlined terms in the first two equations.
Chapter II

PHYSICAL VARIABLES

2.1 Introduction

The equations derived in Chapter I may be considered as prognostic equations for the three components of the velocity vector. In addition to these variables the equations also contain the physical variables of density and pressure. In order to satisfy the elementary, necessary requirement that we have at least as many equations as dependent variables (unknowns) we need additional equations. It is the purpose of this chapter to give the derivations of these equations. In doing so, we shall, for the moment, restrict ourselves to a dry atmosphere. For this case it is sufficient to provide three additional equations which are the gas equation, the thermodynamic equation and the continuity equation. It is most convenient in most problems to consider the six variables to be the three components of \( \vec{v} \), the pressure \( p \), the temperature \( T \) and the density \( \rho \), but the latter variable may naturally be replaced by the specific volume \( \alpha = \rho^{-1} \).

2.2 The Gas Equation for a Perfect Gas

A perfect gas, also called an ideal gas, is one which, by definition, obeys the physical laws of Boyle and Charles. For such a gas it can be shown that the general equation of state, which implies only that there is a unique relation between pressure, density and temperature, is

\[
p = R' \rho T
\]

(2.2.1)

where \( R' \) is the specific gas constant for the perfect gas.

Any of the gases which are part of the dry atmosphere behave, with reasonable approximation, as ideal gases. In order to calculate the numerical value of \( R' \) for a given gas we must make use of Avogadro's law which states that the volume of 1 kg molecular weight of any ideal gas occupies the space of 22.4 m\(^3\) at 273\(^\circ\)K and 1 atm. pressure. Writing (2.2.1) in the form

\[
p\alpha = R' T
\]

(2.2.2)
we may calculate the universal gas constant $R_*$ from (2.2.2) by setting $p = 1013.3 \times 10^2 \text{ N m}^{-2}$ (= 1 atm.), $c = 22.400 \text{ m}^3$ and $T = 273^\circ \text{K}$. We find

$$R_* = 8.3143 \times 10^3 \text{ J kmol}^{-1} \text{ K}^{-1}$$

If the molecular mass of a given ideal gas is $m$, then

$$R' = \frac{R_*}{m} \quad (2.2.3)$$

For example, $R'$ for oxygen is $0.2598 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1} = 259.8 \text{ J kg}^{-1} \text{ K}^{-1}$. The gas constant for nitrogen, the major constituent in the atmosphere, is $R' = 0.2968 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1} = 296.8 \text{ J kg}^{-1} \text{ K}^{-1}$ because the molecular mass is 28.016.

The dry atmosphere is a mixture of perfect gases. Let us consider such a mixture occupying the volume $V$ at temperature $T$. The law of Dalton says that

(a) Each gas occupies the total volume
(b) The gas equation applies for each gas separately
(c) The total pressure is the sum of the partial pressures.

Applying these laws we have for each gas

$$p_q V = M_q R_q T, \quad q = 1, 2, \ldots, Q \quad (2.2.4)$$

where $M_q$ is the mass of constituent $q$, $p_q$ the partial pressure and $R_q$ the gas constant.

Summing all the equations (2.2.4) and using (c) above we get

$$pV = T \left( \sum_{q=1}^{Q} M_q R_q \right) \quad (2.2.5)$$

Dividing (2.2.5) by the total mass $M = \sum_{q=1}^{Q} M_q$ we find

$$\frac{pV}{M} = RT \quad (2.2.6)$$

where

$$R = \frac{1}{M} \sum_{q=1}^{Q} M_q R_q \quad (2.2.7)$$
(2.2.6) is thus the gas equation for the mixture and (2.2.7) gives the formula for the calculation of the gas constant for the mixture. We may in other words consider the mixture as a perfect gas with a gas constant determined by (2.2.7).

R for dry air may be computed from (2.2.7) knowing the composition of the atmosphere and the molecular weights for each of the constituents. One finds

\[ R_d = 287 \, \text{J kg}^{-1} \, \text{K}^{-1} \]

and the "mean molecular weight" of dry air

\[ m_d = \frac{R_n}{R_d} = 28.97 \]

2.3 The First law of Thermodynamics

As an introduction to this topic let us consider one unit of mass (specific volume \( \alpha \)) enclosed in a cylinder, which contains a movable piston (see Figure 2.1). \( d\sigma \) is the area of the cross-section, and it is assumed that the pressure \( p \) is the same inside and outside the cylinder. If the piston moves a distance \( dl \) then the work done by the expanding gas is equal to the product of the force \( (pd\sigma) \) and the distance \( (dl) \). The work is therefore

\[ pd\sigma dl = p \cdot d\sigma \tag{2.3.1} \]

![Figure 2.1](image)

where \( d\sigma = d\varepsilon dl \) is the increase in the volume.

Suppose on the other hand that the piston is fixed. If in that situation we add an amount of energy to the system in the form of heat we can only increase the internal energy, since we prevent the gas from doing any work against the surroundings. If the amount of heat added to the system is \( \delta Q \) we find the increase in temperature from the formula

\[ \delta Q = c_v \, dT \tag{2.3.2} \]
where \( c_v \) is the specific heat at constant volume for the gas. \( c_v \) is therefore defined as the amount of heat which must be added to increase the temperature by one degree.

The general situation is the following, referring again to Figure 2.1 and considering the piston as free to move. If we again consider the addition of an amount of heat \( \delta Q \) it will be used partly to increase the temperature and partly to do work against the surroundings, i.e. the temperature in the volume will increase, and the piston will move. We may write

\[
\delta Q = \delta i + \delta W \tag{2.3.3}
\]

where \( \delta Q \) is the added energy, \( \delta i \) the increase in the internal energy and \( \delta W \) the work done against the surroundings. (2.3.3) expresses only the partitioning of the heat into internal energy and work. The form of \( \delta W \) is \( \delta W = p\delta a \) as seen earlier. An ideal gas is one where the internal energy is a linear function of temperature in addition to having a gas equation of the form (2.2.2). It follows that \( di = c_v \, dT \), and we may therefore write (2.3.3) in the form

\[
\delta Q = c_v \, dT + p\delta a \tag{2.3.4}
\]

which is the first law of thermodynamics for an ideal gas. For such a gas we have also

\[
p\delta a = RT \tag{2.3.5}
\]

or

\[
p\delta a + \delta dp = RdT \tag{2.3.6}
\]

We may therefore also write (2.3.4) in the form

\[
\delta Q = (c_v + R)\,dT - \delta dp \tag{2.3.7}
\]

We note from (2.3.7) that \( (c_v + R)\,dT \) is the total amount of heat if \( dp = 0 \). On the other hand this amount is also \( c_p \,dT \) by definition, where \( c_p \) is the specific heat at constant pressure.

We have therefore

\[
c_p = c_v + R \tag{2.3.9}
\]
In dynamic meteorology we most often need (2.3.4) in the form
\[ H = c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} \tag{2.3.9} \]

obtained by dividing by \( dt \) from (2.3.4). \( H \) is then the amount of heat added per unit mass and per unit time.

For dry air we have \( c_v = 717 \text{ J kg}^{-1} \text{ K}^{-1} \) and therefore \( c_p = 1004 \text{ J kg}^{-1} \text{ K}^{-1} \) since \( R = 287 \text{ J kg}^{-1} \text{ K}^{-1} \). For actual gases we find that \( c_v \) and \( c_p \) are slowly varying functions of temperature. However, for the temperature ranges normally occurring in the atmosphere we may disregard this variation and consider both of them constant.

We note finally that the first law of thermodynamics has been described for a parcel with no motion. The question is, naturally, whether the same law holds for a particle in motion. The answer (not justified here) is yes, provided we include in \( H \) the heat created by dissipation due to the internal work of the frictional forces.

2.4 The Continuity Equation

The continuity equation is a statement expressing the conservation of mass, or in other words: there are no sources and sinks of mass anywhere in the atmosphere. If we consider a volume fixed in space such that the fluid moves through it we may also express the continuity equation in the form that the change in mass inside the volume is equal to the net inflow of mass into the volume. We shall use the latter statement for our derivation.

![Figure 2.2](image)

The volume of the infinitesimal box is \( \delta V = \delta x \delta y \delta z \) and the mass is \( \rho \delta V \). The change per unit time of the mass is \( \frac{\partial \rho}{\partial t} \delta V \), where we must use the local derivative since the box is kept in the same location.
We shall now calculate the net-inflow. If 1 is a point in the centre of the left side of the co-ordinates \((x - \frac{1}{2} \delta x, y, z)\) and 2 has the co-ordinates \((x + \frac{1}{2} \delta x, y, z)\) we find that the net inflow in the x-direction is

\[
\rho_1 u_1 \delta y \delta z - \rho_2 u_2 \delta y \delta z =
\]

\[
= \left( p_o u_o \, \frac{1}{2} \left( \frac{\partial (pu)}{\partial x} \right)_o \delta x \right) \delta y \delta z - \left( p_o u_o + \frac{1}{2} \left( \frac{\partial (pu)}{\partial x} \right)_o \delta x \right) \delta y \delta z =
\]

\[
= - \left( \frac{\partial (pv)}{\partial x} \right)_o \delta V
\]

The net inflow in the y- and z-directions are, respectively,

\[
- \left( \frac{\partial (pv)}{\partial y} \right)_o \delta V, \text{ and } - \left( \frac{\partial (pw)}{\partial z} \right)_o \delta V
\]

The continuity equation is therefore

\[
\frac{\partial p}{\partial t} = - \left[ \frac{\partial pu}{\partial x} + \frac{\partial pv}{\partial y} + \frac{\partial pw}{\partial z} \right]
\tag{2.4.1}
\]

We may write (2.4.1) in the short form

\[
\frac{\partial p}{\partial t} = - \nabla \cdot (pv)
\tag{2.4.2}
\]

From (2.4.1) or (2.4.2) it is seen that we may obtain the alternate forms

\[
\frac{\partial p}{\partial t} + \nabla \cdot \nabla p + \rho \nabla \cdot \nabla = 0
\tag{2.4.3}
\]

or

\[
\frac{\partial p}{\partial t} + \rho \nabla \cdot \nabla = 0
\tag{2.4.4}
\]

2.5 Some Special Considerations

The three additional equations, derived in 2.2 - 2.4 are quite general for the atmosphere, because the assumption that the atmosphere behaves as a perfect gas is satisfied within reasonable limits. For certain processes it is permissible to make special assumptions. The most important of these special assumptions is probably the adiabatic assumption where it is assumed that \(H = 0\) for the process. It is obvious that this assumption can be made whenever the motion is so fast that the heat exchange between the particle and the surroundings is negligible.
For the adiabatic motion therefore, we have
\[ c_v \, dT + p \, da = 0 \]  
(2.5.1)

or
\[ c_p \, dT - a \, dp = 0 \]  
(2.5.2)

Substituting from the gas equation in (2.5.1) we find after a simple integration
\[ \frac{T}{T_1} = \left( \frac{a}{a_1} \right)^{\frac{c_v}{R}} \]  
(2.5.3)

where \( T_1 \) and \( a_1 \) are corresponding values. (2.5.3) gives the relation between temperature and specific volume in an adiabatic process, and it is seen that if \( T \) increases, \( a \) will decrease and vice versa. A compression in an adiabatic process is therefore accompanied by an increase of temperature, while expansion is associated with a decrease of temperature.

On the other hand, substituting from the gas equation in (2.5.2) we get after integration
\[ \frac{T}{T_1} = \left( \frac{p}{p_1} \right)^{\frac{c_p}{R}} \]  
(2.5.4)

From (2.5.4) we note that an increase in pressure is accompanied by an increase in temperature. Equating the right-hand sides of (2.5.3) and (2.5.4) we find
\[ \frac{a}{a_1} = \left( \frac{p}{p_1} \right)^{\frac{c_v}{c_p}} \]  
(2.5.5)

which shows that an increase in \( p \) corresponds to a decrease in \( a \) and vice versa.

It is sometimes found convenient to describe a thermodynamic process by a curve in a diagram having \( a \) as abscissa and \( p \) as ordinate. Both diabatic (\( R \neq 0 \)) and adiabatic (\( R = 0 \)) processes may be illustrated in this way. From (2.5.5) we find that an adiabatic process is described by a curve having the equation
\[ p = \text{const.} \, a^{-\frac{c_p}{c_v}} \]  
(2.5.6)
Since $c_p/c_v > 1$ we find that this curve will have a larger negative slope than a common hyperbola. The latter curve is found for an isothermal process (i.e. a process where the temperature remains constant). From the gas equation we find for an isothermal process

$$p = (RT) \cdot a^{-1} \quad (2.5.7)$$

where $T$ is a constant for the process.

The relations (2.5.6) and (2.5.7) are illustrated in Figure 2.3, which also shows the curves for an isobaric ($p = \text{const.}$) process and an isosteric ($\alpha = \text{const.}$) process.

![Diagram of isosteric, isobaric, adiabatic, and isothermal processes](image)

Figure 2.3

It is often of interest to find a conservative quantity for a given process, if such an invariant exists. This is the case for the adiabatic process. Suppose that we select $p_1 = p_o = 1000$ mb in (2.5.4) and denote the corresponding temperature $\theta$. We have then

$$\theta = T \left( \frac{p}{p_0} \right)^{c_p} \quad (2.5.8)$$

which we may consider as the definition of the potential temperature $\theta$ which is the temperature a particle would have if it was brought adiabatically to the pressure $p_o = 1000$ mb. It is obvious that $\theta$ is a conservative quantity for a given adiabatic process.

We may, however, also consider the potential temperature in the following
way: for a given particle with temperature \( T \) and pressure \( p \) there exists a potential temperature which can be computed from (2.5.8). If no heat is added to the particle the potential temperature will remain the same even if the particle moves to another position with a new pressure. The new temperature can in that case be computed from (2.5.8). However, if heat is added by the amount \( H \) per unit mass and unit time, then the potential temperature will change. How can we calculate the change in the potential temperature? We find the results by a differentiation of (2.5.8) with respect to time to obtain

\[
\frac{1}{\theta} \frac{d\theta}{dt} = \frac{1}{T} \frac{dT}{dt} - \frac{R}{c_p} \frac{1}{p} \frac{dp}{dt}
\]  

(2.5.9)

or

\[
c_p \frac{T}{\theta} \frac{d\theta}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt}
\]  

(2.5.10)

A comparison of (2.5.10) and the latter part of (2.3.9) shows that (2.5.10) can be written in the form

\[
\frac{d(\ln \theta)}{dt} = \frac{1}{c_p} \frac{H}{T}
\]  

(2.5.11)

This form of the thermodynamic equation will be useful later.
Chapter III

THE PREDICTION PROBLEM

3.1 The Complete System of Equations

The equations of motion, considered in Chapter I, and the three equations for the physical variables, treated in Chapter II, form together a system of 6 equations for the 6 dependent variables: u, v, w, p, ρ and T. In meteorology it is normally an advantage to consider the equations in their Eulerian form where particle derivatives are replaced by the local derivatives plus the convective accelerations. As seen from the previous chapters we then obtain equations which are of the first order with respect to time in those equations which are differential equations. The only exception to this is the gas equation which is very straightforward.

We note furthermore that the thermodynamic energy equation (2.3.9) is the only equation which contains more than one time derivative. This situation may be changed by eliminating one of the time derivatives using the gas equation and the continuity equation. Thus from the form in (2.3.9) we obtain using the gas equation

\[ H = \frac{c_p}{R} \left[ \frac{d}{dt} \rho + \rho \frac{d}{dt} \right] - a \frac{d}{dt} \]

(3.1.1)

From the relation \( a = \rho^{-1} \) we may write (3.1.1) in the form

\[ H = \frac{c_p}{R} \left[ \frac{1}{\rho} \frac{d}{dt} \rho - \frac{\rho}{\rho^2} \frac{d}{dt} \right] - \frac{1}{\rho} \frac{d}{dt} \]

(3.1.2)

The expression for \( \frac{d}{dt} \rho \) in the continuity equation (2.4.4) is substituted in (3.1.2), and we get

\[ H = \frac{c_v}{R} \frac{1}{\rho} \frac{d}{dt} \rho + \frac{c_p}{R} \rho \mathbf{v} \cdot \mathbf{v} \]

(3.1.3)

which may be rearranged in the form

\[ \frac{d}{dt} \rho = - \frac{c_p}{c_v} \rho \mathbf{v} \cdot \mathbf{v} + \frac{R}{c_v} H \rho \]

(3.1.4)
Having obtained (3.1.4) we may now summarise the prognostic equations in the following forms

\[ \frac{du}{dt} = - \nabla \cdot \nabla u - \frac{1}{\rho} \frac{dp}{dx} + f v - e w + \frac{1}{\rho} \rho x \]

\[ \frac{dv}{dt} = - \nabla \cdot \nabla v - \frac{1}{\rho} \frac{dp}{dy} - f u + \frac{1}{\rho} \rho y \]

\[ \frac{dw}{dt} = - \nabla \cdot \nabla w - \frac{1}{\rho} \frac{dp}{dz} + e u - g + \frac{1}{\rho} \rho z \quad (3.1.5) \]

\[ \frac{dp}{dt} = - \nabla \cdot \nabla p - \rho \nabla \cdot \nabla \]

\[ \frac{dp}{dt} = - \nabla \cdot \nabla p - \frac{c_p}{c_v} \rho v \cdot \nabla + \frac{R}{c_v} \rho H \]

The system (3.1.5) is the set of prognostic equations which must be used in a general prediction problem. The five variables in the system are u, v, w, p, ρ, while the temperature does not appear in the system. However, if desired, the temperature can always be computed from the gas equation knowing p and ρ.

Solutions to the general system (3.1.5) in a closed mathematical form are unknown in the general case because of the mathematical complexity of the equations. If we were to obtain a particular solution it has to be obtained in an approximative way by numerical methods. Such a solution is, however, obtainable in principle if we imagine that we have observations of u, v, w, p and ρ in a sufficiently fine network of points which covers the whole atmosphere as a three-dimensional grid. The initial state would have to be supplemented by a set of boundary conditions (see section 3.2) but these can be formulated. In spite of the method, outlined above as possible in principle, we have not so far obtained numerical solutions to (3.1.5). One reason for this is that we do not make observations in the atmosphere which make it possible to use the system. The present methods of observation give us only the horizontal wind \( \mathbf{v}_2 \) and not w. In addition, we make use of the hydrostatic equation in order to obtain the physical variables p, ρ and T.

Because of the use of the hydrostatic equation it appears that the prediction problem is quite different from that formulated in (3.1.5). The difficulty is immediately apparent. If we replace the third equation in (3.1.5) by the hydrostatic equation. In that case we have first of all lost the prognostic equation for w,
but, in addition, if we predict new values of $p$ and $p$ from the fourth and fifth equations we have no guarantee that they will be in hydrostatic equilibrium. We shall consider the solution to these problems in section 3.3.

3.2 Boundary Conditions and Initial Values

In order to obtain solutions of an analytical or numerical nature to systems such as (3.1.5) or any other system describing atmospheric motion we must formulate a set of boundary conditions which the solution must satisfy. We distinguish between kinematical and dynamical boundary conditions.

We shall first consider the kinematical boundary conditions. If the boundary is a rigid surface at rest (such as the earth's surface) we must require that the normal component is zero, or that the motion is tangential to the surface. If the surface has the equation

$$f(x,y,z) = 0$$

(3.2.1)

we can express the boundary condition in the form

$$\nabla \cdot \nabla f = 0$$

(3.2.2)

since $\nabla f$ is directed along the normal to the surface.

A boundary condition of this kind is always used in numerical prediction problems along the surface of the earth, although it is normally formulated in a slightly different form. Suppose that the height of the surface over mean sea-level is given by

$$z = h(x,y)$$

(3.2.3)

where $z$ is the height of the topography (mountains). We have then

$$f(x,y,z) = z - h(x,y) = 0$$

(3.2.4)

and the condition (3.2.2) becomes

$$- u \frac{\partial h}{\partial x} - v \frac{\partial h}{\partial y} + w = 0$$
or

\[ w = \vec{v} \cdot \vec{v}_h \]  

(3.2.5)

where \( w \) is the vertical velocity created by the sloping terrain.

If the boundary is in motion, such as an internal surface of discontinuity, we must require that the velocity components normal to the internal boundary are the same on either side. If the equation for the moving surface of discontinuity is

\[ f(x, y, z, t) = 0 \]  

(3.2.6)

and \( \vec{v}_1 \) and \( \vec{v}_2 \) are the velocity vectors on the two sides of the discontinuity surface, we get for the boundary condition

\[ \vec{v}_1 \cdot \vec{v}_f = \vec{v}_2 \cdot \vec{v}_f \]  

(3.2.7)

The left side (see Figure 3.1) is \( V_1 |\vec{v}_f| \cos \theta_1 \) where \( \theta_1 \) is the angle between the vectors \( \vec{v}_1 \) and \( \vec{v}_f \). \( V_1 \cos \theta_1 \) is the component normal to the surface. We may also write \( \vec{v}_1 \cdot \vec{v}_f = \vec{v}_{1n} |\vec{v}_f| \). In a similar way we get for the second fluid \( \vec{v}_2 \cdot \vec{v}_f = \vec{v}_{2n} |\vec{v}_f| \cos \theta_2 \). It is thus obvious that (3.2.7) expresses that the components normal to the discontinuity surface are the same in the two fluids.

The conditions mentioned above are acceptable for the case with no friction. If friction is considered in the problem, we must also require that the tangential components along a rigid, fixed surface are zero, and that the tangential components are the same in the two fluids along a moving surface of discontinuity. If these conditions were not fulfilled, and we permitted a finite difference between tangential components for two particles very close to each other, it would result in infinitely large stresses.

![Figure 3.1](image-url)
The dynamic boundary condition at an internal discontinuity surface is that pressure is continuous across the interface. If this is not the case we would get infinitely large pressure forces. Note, however, that density and temperature may be discontinuous.

In the description of the numerical solution of (3.1.5) we have inherently considered the prediction problem as an initial value problem where we know the atmospheric variables at the time $t = 0$, and we can then, using equations (3.1.5) and the boundary conditions mentioned in this section, compute the values of the dependent variables at a later time $\Delta t$, at which time we repeat the whole process. It is seen from (3.1.5) that such a marching problem is possible in principle, although we shall not consider here the details of the necessary finite difference system in space and time. We shall, however, stress one practical difficulty which is unavoidable.

The system (3.1.5) was derived by a consideration of the physical laws (Newton's second law, the conservation of mass, the first law of thermodynamics and the gas equation) for a particle in the atmosphere. In practice, we do not know the initial state for each particle in the atmosphere, but the initial state is obtained by an interpolation using the observed data in an irregularly arranged network of observing stations separated by many kilometres. It is thus obvious that the initial state is at best a smoother motion on a relatively large scale. The system of equations (3.1.5) may not apply to the "observed" large-scale initial state, and it may have to be modified. This question will be considered in Chapter XIII.

3.3 The Quasi-static Prediction Problem

In this section we shall investigate some of the implications of the replacement of the third equation of motion by the hydrostatic equation. The main question concerns the vertical velocity. One may, at first sight, gain the impression that it is necessary to disregard the vertical velocity altogether because of the severe constraints imposed by the balance of the vertical component of the pressure force and the acceleration of gravity. If this were the case, it is obvious that the hydrostatic assumption would have caused too severe distortion to be of any usefulness in prediction problems.

The solution to the problem is that the vertical velocity can be computed by a diagnostic equation from a knowledge of the horizontal wind and the physical variables, pressure and density. Before deriving the equation from which the vertical velocity
can be computed we shall first consider some of the implications of the result.

We stress that the result is obtained by requiring that the hydrostatic relation is satisfied at all points and at all times. The calculated vertical velocity can therefore be characterized as the one which must exist in the atmosphere in order to maintain hydrostatic equilibrium in all places. This vertical velocity is normally denoted \( w_R \), where the subscript \( R \) refers to L. F. Richardson who first derived the equation.

The technique of deriving Richardson's equation is based on obtaining two pressure tendency equations. The first of these is already included as the last equation in (3.1.5). The second equation is obtained as follows. From a time-differentiation of the hydrostatic equation we get

\[
\frac{\partial}{\partial z} \left( \frac{\partial p}{\partial t} \right) = -g \frac{\partial p}{\partial t}
\]

(3.3.1)

Substituting from the continuity equation we may write (3.3.1) in the form

\[
\frac{\partial}{\partial z} \left( \frac{\partial p}{\partial t} \right) = g \nabla_2 \cdot (p \nabla) + g \frac{\partial p w_R}{\partial z}
\]

(3.3.2)

where we have divided the three-dimensional mass divergence in the horizontal and vertical parts. Equation (3.3.2) may now be integrated from an arbitrary height \( z \) to the top of the atmosphere \( (z \to \infty) \). At the upper boundary we have \( \partial p/\partial t \to 0 \), \( \rho w \to 0 \) as \( z \to \infty \). The result is therefore

\[
\frac{\partial p}{\partial t} = -g \int_z^{\infty} \nabla_2 \cdot (p \nabla) \, dz + g \rho w_R
\]

(3.3.3)

which is the pressure tendency equation. It says simply that the change in pressure at the bottom of an atmospheric column rising from \( z \) to \( \infty \) is equal to the weight of the net inflow of mass. The first term measures the integrated effect of the horizontal mass convergence while the last term measures the vertical mass flux through the bottom of the column.

The final equation in (3.1.5) may now be written in the following form where we have divided the three-dimensional operators into their horizontal and vertical parts

\[
\frac{\partial p}{\partial t} = -\nabla_2 \cdot \nabla_2 p + w_R \rho p - \frac{c_p}{c_v} p \nabla_2 \cdot \nabla - \frac{c_p}{c_v} p \frac{\partial w_R}{\partial z} + \frac{R}{c_v} \rho H
\]

(3.3.4)
The second term on the right-hand side of (3.3.4) arises from \(- w(\partial p/\partial z)\) into which we have introduced the hydrostatic relationship. Equating the right-hand sides of (3.3.3) and (3.3.4) we get

\[
\frac{dW_R}{dz} = \frac{c_v}{c_p} \frac{1}{p} \left[ - \nabla_v \cdot \nabla_p - \frac{c_p}{c_v} \mathbf{v} \cdot \nabla + \frac{R}{c_v} \rho H + g \int_0^Z \nabla_v \cdot (\rho \mathbf{v}) \, dz \right]
\]  

(3.3.5)

It can be seen that the right-hand side of (3.3.5) can be evaluated from observations of \(\nabla_v\), \(p\) and \(\rho\) at a given time. We note that we must know the distribution of the variables not only at the point where we want \(w\) but also in the immediate neighbourhood in order to evaluate such quantities as \(\nabla_v p\), \(\nabla_v \cdot \nabla\) and \(\nabla_v \cdot (\rho \mathbf{v})\) in the whole column above the point in question. The value of \(w\) itself can be obtained from (3.3.5) by integration from the earth surface to the height \(z\).

Denoting the complete right-hand side of (3.3.5) by \(A = A(x,y,z)\) we find

\[
w_R = w(x,y,z) = w(x,y,z_0) + \int_{z_0}^{z} A(x,y,z') \, dz'
\]  

(3.3.6)

where \(z'\) is the dummy variable for the integration and \(w_0 = w(x,y,z_0)\) is the vertical velocity at \(z = z_0(x,y)\) which is the height of the topography. Equation (3.3.6) shows that \(w_R\) depends on data in the whole vertical column.

The equation above is Richardson's equation for the computation of the vertical velocity \(w_R\) which is consistent with the hydrostatic relation. It replaces the third equation of motion in determining the vertical velocity. The quasi-static system of equations which may be used to predict the future state of the atmosphere may therefore be written in the form

\[
\begin{align*}
\frac{du}{dt} &= - \nabla \cdot \mathbf{v}u - \frac{1}{p} \frac{\partial p}{\partial x} + f v + \frac{1}{p} F_x \\
\frac{dv}{dt} &= - \nabla \cdot \mathbf{v}v - \frac{1}{p} \frac{\partial p}{\partial y} - f u + \frac{1}{p} F_y \\
\frac{\partial p}{\partial t} &= - \nabla \cdot \mathbf{v}p - \frac{c_p}{c_v} \mathbf{v} \cdot \nabla v + \frac{R}{c_v} \rho H \\
\rho &= - \frac{1}{g} \frac{\partial p}{\partial z} \\
w_R &= w_0 + \int_{z_0}^{z} A(x,y,z') \, dz'
\end{align*}
\]  

(3.3.7)
where we now have three prognostic equations for $u$, $v$ and $p$, and two diagnostic
equations for $p$ and $w_R$. This system is essentially the system proposed and used by
L. E. Richardson in his famous pioneering numerical prediction experiment, but it
is also used nowadays in some general circulation experiments.

In this section we have given the derivations in essentially the same way as
they were given by Richardson where we have used height as the vertical co-ordinate.
However, the physical content of the system (3.3.7) is the same as is used in most
prediction schemes which are based on the so-called primitive equations by which we
mean the system (3.3.7) or the equivalent system with another vertical co-ordinate.
The set of primitive equations is therefore the same as we have called the
quasi-static equations here. In a later section (Chapter V) we shall consider the
transformations to other vertical co-ordinates.
Chapter IV

STATIC EQUILIBRIUM IN THE ATMOSPHERE

4.1 The Basic Equations for Equilibrium

Let us consider the state where all particles are and remain at rest relative to the earth, i.e. \( \vec{v}_x = 0 \). From the vector equation for the relative motion we find that since \( d\vec{v}_x/dt = 0 \) it follows that

\[
0 = -\frac{1}{\rho} \vec{v}_p + \vec{g}
\]

(4.1.1)

because the Coriolis force and the frictional force both depend on the velocity and disappear when the velocity disappears.

In the usual co-ordinate system we find

\[
\vec{g} = -g\vec{k} = -\vec{v}\phi
\]

(4.1.2)

where \( \phi \) is the geopotential, and we may write (4.1.1) in the form

\[
\vec{v}\phi = -\nabla \phi
\]

(4.1.3)

For two arbitrary points 1 \((x, y, z)\) and 2 \((x+dx, y+dy, z+dz)\), when the vector from 1 to 2 is denoted \( \vec{dr} \), we have

\[
d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}
\]

(4.1.4)

Also, we found earlier (see 1.7.7) that

\[
db = b_2 - b_1 = \vec{v}b \cdot \vec{dr}
\]

(4.1.5)

where \( b \) is an arbitrary scalar. Combining (4.1.4) and (4.1.5) we have

\[
d\phi = -\nabla \phi
\]

(4.1.6)
The equiscalar surfaces for $\phi$ are called geopotential surfaces, while the equiscalar surfaces for $p$ are the isobaric surfaces. The geopotential surfaces are not identical with constant height surfaces where the height is measured relative to mean sea-level. The reason is that $g$ increases somewhat from pole to equator, and the geopotential surfaces slope "downward" towards the poles.

![Diagram](image)

**Figure 4.1**

It follows from (4.1.3) that the isobaric and the geopotential surfaces coincide in the case of static equilibrium. Consider now two pair of points along different normals to two geopotential surfaces (see Figure 4.1)

Since the two surfaces are geopotential surfaces we have

$$d\Phi_A = \Phi(a') - \Phi(a) = \Phi(b') - \Phi(b) = d\Phi_B \quad (4.1.7)$$

However, the two surfaces are also isobaric surfaces. We have therefore

$$dp_A = p(a') - p(a) = p(b') - p(b) \propto dp_B \quad (4.1.8)$$

It follows then from (4.1.6) that

$$a_A = a_B \quad (4.1.9)$$

and it follows that the isosteric surfaces coincide with the geopotential (and isobaric) surfaces in the case of static equilibrium.

It is now easy to see that the isothermal surfaces coincide with the geopotential surfaces in the equilibrium case. The reason is that, within a geopotential surface, we have a constant pressure and a constant specific volume. The gas equation tells us that the temperature is then also constant at the surface.
Since the potential temperature is a unique function of temperature and pressure (see 2.5.3) we find that the geopotential surface is also a surface of constant potential temperature, i.e. an isentropic surface.

In summary, we may say that in the case of no relative motion we have coincidence between the geopotential, isobaric, isopycnic, isothermal and isentropic surfaces.

When the atmosphere is in motion ($\vec{v}_r \neq 0$) as is always the case we have different slopes to all the surfaces mentioned above, and it follows that the case of static equilibrium is completely artificial. However, the case serves as a simple example of the stratification of the atmosphere, and it is important in many investigations of a theoretical nature where the motion can be disregarded altogether or at least considered to be small.

4.2 Static Equilibrium and Hydrostatic Balance

It is seen from (4.1.3) that if we have static equilibrium ($\vec{v}_r = 0$), we have also the hydrostatic relation

$$\frac{dp}{dz} = -g \beta$$

(4.2.1)

because (4.2.1) is the component of (4.1.3) along the vertical unit vector which is opposite to $\vec{g}$. It follows naturally also from a direct consideration of the third equation of motion that (4.2.1) is only exact when $\vec{v}_r = 0$. Therefore, when we use (4.2.1) in evaluating atmospheric observations and in prediction problems (see Chapter III), it is obviously an approximation, but as demonstrated at the end of Chapter I an extremely accurate one, since the neglected terms in large-scale motion are more than two orders of magnitude smaller than $g$.

4.3 Hydrostatic Calculations

The practical use of (4.2.1) in the atmosphere is obvious in evaluating the geopotential $\phi = gz$, where $g$ is considered to be a constant equal to 9.8 m s$^{-2}$. Since the measured quantities are pressure, temperature and humidity, it is an advantage to write the hydrostatic relation in terms of these quantities. The first step is to replace the actual temperature by the virtual temperature (see Physical Meteorology (section 6.8)). We shall in the following assume that this has been done in all cases. However, we shall maintain the notation $T$ for the virtual temperature.
and avoid the subscript v. We get then from (4.2.1)

\[ dz = - \frac{R}{g} \frac{T}{p} \, dp \]  
(4.3.1)

using the gas equation. An integration of (4.3.1) leads to the result

\[ z_2 = z_1 - \frac{R}{g} \int_{p_1}^{p_2} T d(ln \, p) \]  
(4.3.2)

In practice where we have corresponding values of T and p at irregular points, it is necessary to carry out the integration by numerical or graphical methods. If the latter method is used, we make use of so-called aerological diagrams which are diagrams which normally have pressure (or a function of pressure) as ordinate, and a function of temperature and pressure as abscissa. We shall not enter into a discussion here of the details of the integration (4.3.2) or of the aerological (or thermodynamic) diagrams.

If we define \( T_m \) as the mean temperature with respect to the logarithm of pressure i.e.

\[ T_m = \frac{1}{\ln \left( \frac{p_2}{p_1} \right)} \int_{p_1}^{p_2} T d(ln \, p) \]  
(4.3.3)

we may write (4.3.2) in the form

\[ z_2 = z_1 + \frac{R}{g} T_m \ln \left( \frac{p_1}{p_2} \right) \]  
(4.3.4)

In meteorological practice we have an analysis of z in the various isobaric surfaces. The difference \( h = z_2 - z_1 \), is called the relative geopotential or the thickness of the layer between the two isobaric surfaces \( p_1 \) and \( p_2 \). It is seen from (4.3.4) that the thickness is proportional to the mean temperature \( T_m \).

It is naturally also possible to use the hydrostatic relation to calculate the pressure difference corresponding to a given geopotential difference. We find from (4.3.1) that

\[ \frac{dp}{p} = - \frac{g}{R} \frac{1}{T} \, dz \]  
(4.3.5)
and therefore

\[ \ln \left( \frac{P_1}{P_2} \right) = \frac{g}{R} \int_{z_1}^{z_2} \frac{dz}{T} \]  

(4.3.6)

from which we can calculate the pressure, when \( z_1 \) and \( z_2 \) are given, and \( T \) is known as a function of height, \( T = T(z) \).

(4.3.1) can be used directly in cases where \( dp \) and \( dz \) are small. As an example we may mentioned that if a surface pressure map is given, we are often interested in converting this map to the geopotential field of the isobaric surface, \( p = 1000 \text{ mb} \). This can be done in an approximative way using (4.3.1) with \( \delta p = -5 \text{ mb} \) (The interval for the isolars on the surface map), \( p = 1000 \text{ mb}, R = 287 \text{ m}^2 \text{s}^{-2} \text{ deg}^{-1}, g = 9.8 \text{ m} \text{s}^{-2} \), and we find \( dz \approx 40 \text{ m} \) when \( T = 273^0 \text{K} \). If this calculation is of sufficient accuracy, it is therefore possible to make the desired conversion by refiling the isolars as isolines of the geopotential using the fact that a difference in 5 mb corresponds to a height difference of 40 m. In order to investigate the accuracy of such a simple conversion formula we may proceed as follows. Suppose that the temperature was \( T_0 + \delta T \), where \( T_0 = 273^0 \text{K} \) and that the pressure was \( p_0 + \delta p \), where \( p_0 = 1000 \text{ mb} \). We will then get a height which may be written \( dz_0 + \delta(dz) \) where \( dz_0 \) is the height difference computed for the pressure difference \( dp \), \( p = p_0 \) and \( T = T_0 \). We find from (4.3.1)

\[ \delta(dz) = -\frac{R}{g} \delta p \left( \frac{1}{p_0} \delta T - T_0 \delta p \right) \]  

(4.3.7)

or

\[ \delta(dz) = +0.40 \delta p - 0.15 \delta T \]  

(4.3.8)

\( \delta p \) may in practice be as large as 40 mb. In that case we find \( \delta(dz) = 1.6 \text{ m} \) if \( \delta T = 0 \). On the other hand, if \( \delta T = 30^0 \), say, and \( \delta p = 0 \), we find \( \delta(dz) = -4.5 \text{ m} \). It seems therefore that the error \( \delta(dz) \) in the calculation may be quite large. However, in most cases, we have a compensation in (4.3.8) since \( \delta p \) and \( \delta T \) often have the same sign because centres of high pressure are warm and centres of low pressure are cold.

4.4 Examples of Static Atmospheres

In theoretical considerations and in order of magnitude estimations it is
often advantageous to use rather simple distributions of the atmospheric variables, in particular the variations with height of pressure, temperature, density, etc. A number of simple atmospheric models have been constructed for such purposes. They are characterized by their simplicity and by the fact that the hydrostatic equation can be integrated exactly using simple functions. However, the model atmospheres are not normally realistic compared with the real atmosphere. We shall consider some examples.

\( \text{(c) The homogeneous atmosphere} \)

The density is equal to a constant \( p_o \) in this atmosphere and independent of space and time. We find from the hydrostatic equation that

\[
p = p_o - \frac{g}{\rho_o} z
\]  

(4.4.1)

where \( p_o \) is the pressure for \( z = 0 \). The homogeneous atmosphere has a finite height \( H \) calculated from (4.4.1) by setting \( z = H \) for \( p = 0 \). We find

\[
H = \frac{p_o}{\frac{g}{\rho_o}}
\]  

(4.4.2)

We may define a temperature in the homogeneous atmosphere from the gas equation

\[
T = \frac{1}{\rho_o} \frac{1}{R} p
\]  

(4.4.3)

and we find

\[
T - \frac{p_o}{R \rho_o} - \frac{g}{R} z = T_0 - \frac{g}{R} z
\]  

(4.4.4)

where

\[
T_0 = \frac{p_o}{R \rho_o}
\]  

(4.4.5)

(4.4.4) shows that the temperature decreases linearly with height in a homogeneous atmosphere. (4.4.2) may also be written

\[
H = \frac{RT_0}{g}
\]  

(4.4.6)
From (4.4.6) we find that \( H = 8000 \text{ m} \) if \( T_0 = 273^\circ \text{K} \). This value of \( H \) is often used in numerical estimates. The lapse rate \( (\gamma = -\frac{dT}{dz}) \) of the homogeneous atmosphere is \( g/R = 3.4^\circ \text{K}/100 \text{ m} \).

(b) **The isothermal atmosphere**

In this atmosphere we have \( T = T_0 = \text{const} \). From the hydrostatic equation we get by integration

\[
p = p_0 e^{-\frac{g}{R T_0} z}
\]

(4.4.7)

(4.4.7) shows that the isothermal atmosphere is of infinite extent because \( p \to 0 \) when \( z \to \infty \). The scale height for an isothermal atmosphere is often defined as the height at which the pressure has decreased to \( e^{-1} \) of the surface pressure. We find that

\[
H_S = \frac{RT_0}{g} = H
\]

(4.4.8)

or, that the scale height is equal to the height of the homogeneous atmosphere having the same surface temperature as the isothermal atmosphere.

The density in the isothermal atmosphere can be calculated from the gas equation, and we find

\[
\rho = \rho_0 e^{-\frac{g}{R T_0} z}
\]

(4.4.9)

when \( \rho_0 = p_0/(RT_0) \).

(c) **The atmosphere with constant lapse rate**

The lapse rate \( \gamma \) is defined as \( -(dT/dz) \). The temperature is then

\[
T = T_0 - \gamma z
\]

(4.4.10)

when \( T_0 \) is the temperature for \( z = 0 \). The hydrostatic equation gives

\[
dp = -\frac{g}{R(T_0 - \gamma z)} \, dz
\]

(4.4.11)
or
\[
\frac{dp}{p} = g \frac{1}{R} \frac{1}{\gamma} \frac{dT}{T_o - \gamma z} = \frac{g}{R} \frac{1}{\gamma} \frac{dT}{T}
\]  
(4.4.12)

or by integration
\[
p = p_o \left( \frac{T}{T_o} \right)^{\frac{g}{R} \frac{1}{\gamma}}
\]  
(4.4.13)

which gives \( p \) as a function of \( T \), and thereby \( p = p(z) \).

For the density we find
\[
\rho = \frac{p}{RT} = \frac{p_o}{RT_o} \left( \frac{T}{T_o} \right)^{-\frac{g}{R} \frac{1}{\gamma} - 1} = \rho_o \left( \frac{T}{T_o} \right)^{-\frac{g}{R} \frac{1}{\gamma} - 1}
\]  
(4.4.14)

It is seen from (4.4.10) that the constant lapse rate atmosphere has a finite \( H = H_{\gamma} \) defined by
\[
H_{\gamma} = \frac{T_o}{\gamma}
\]  
(4.4.15)

For \( \gamma = 0.65 \times 10^{-2} \text{ K m}^{-1} \) we find for \( T_o = 273 \text{ K} \) that \( H_{\gamma} = 4.2 \times 10^4 \text{ m} = 42 \text{ km} \). At \( z = H_{\gamma} \) we have \( T = 0 \), \( p = 0 \) and \( \rho = 0 \).

(a) **The Adiabatic Atmosphere**

This atmosphere is defined as having a constant potential temperature \( \theta = \theta_o \). From the definition of potential temperature we find
\[
\frac{1}{\theta} \frac{d\theta}{dz} = \frac{1}{T} \frac{dT}{dz} - \frac{R}{c_p} \frac{1}{p} \frac{dp}{dz}
\]  
(4.4.16)

Using the hydrostatic equation we may write (4.4.16) in the form
\[
\frac{1}{T} \left( \frac{dT}{dz} + \frac{g}{c_p} \right) = 0
\]  
(4.4.17)

which says that the adiabatic atmosphere is characterized by a constant lapse rate
\[
\gamma = -\frac{dT}{dz} = -\frac{g}{c_p}
\]  
(4.4.18)
The formulae for the adiabatic atmosphere can therefore be obtained from those for the constant lapse rate atmosphere by setting $\gamma = \gamma_a = \frac{6}{c_p} = 0.01$. The height of the adiabatic atmosphere becomes $H_{\gamma_a} = 27.3 \text{ km}$.

(c) **Standard atmosphere**

None of the atmospheric static models analysed under (c)-(d) is a good approximation to the real atmosphere in its entirety. A number of standard atmospheres consisting of several layers, each characterized by one of the models analysed here, has been constructed. As a relatively simple example considering only the troposphere and stratosphere we may mention a model where the lowest 10 km are characterized by a constant lapse rate layer with $\gamma = 0.65^\circ \text{K/100 m}$ and an isothermal layer on top of this layer. The lower layer is a simple model of the troposphere, while the upper layer is a representation of the stratosphere. The boundary between the two layers is called the tropopause.
Chapter V

PRESSURE AS VERTICAL CO-ORDINATE

5.1 Introduction

The equations derived in Chapters I and II as well as the general prediction problem considered in Chapter III have used a system with height \( z \) as the vertical coordinate. Under the assumption of hydrostatic equilibrium it becomes advantageous in many problems to introduce other vertical co-ordinates. The most common vertical coordinate in practical work in weather analysis and prediction is pressure, but density, temperature or potential temperature are also used occasionally. In the following sections we shall derive the equations using pressure as the vertical coordinate.

5.2 General Considerations

The hydrostatic equation discussed at length in Chapter IV provides a unique and monotonic relationship between height and pressure when the vertical stratification is given, for example, by a set of observed values of \( T \) and \( p \). The integration of the hydrostatic equation then gives height as a function of \( T \) and \( p \) as shown in (4.3.2) and the later examples. It is in fact such an integration which is carried out using observed values of \( T \) and \( p \) obtained from a radiosonde ascent. We may, in particular, obtain the height of selected isobaric levels over the station from such an integration.

In the earlier chapters we have considered the kinematic and physical variables in the atmosphere as functions of space and time, i.e. as functions of \( x, y, z \) and \( t \). In view of the hydrostatic equation we may equally well consider these variables as functions of \( x, y, p \) and \( t \). While pressure is a dependent variable \( p = p(x,y,z,t) \) in the first system, it becomes an independent variable in the second. The height \( z \) is an independent variable in the first system, but becomes a dependent variable \( z = z(x,y,p,t) \) in the second.
5.3 Individual and Local Time Derivatives

Let us consider an arbitrary variable \( b = b(x, y, p, t) \). Let a particle at \( t = t_0 \) be in the "point" \( (x_0, y_0, p_0) \). At the time \( t = t_0 + dt \) the particle is in the point \( (x_0 + dx, y_0 + dy, p_0 + dp) \). In analogy to the development in section 1.7 we may write the change \( db = b(x_0 + dx, y_0 + dy, p_0 + dp, t_0 + dt) - b(x_0, y_0, p_0, t_0) \) in the form

\[
\frac{db}{dt} = \left(\frac{\partial b}{\partial x}\right)_p dx + \left(\frac{\partial b}{\partial y}\right)_p dy + \left(\frac{\partial b}{\partial p}\right)_p dp + \left(\frac{\partial b}{\partial t}\right)_p dt + \text{h.o.t.} \tag{5.3.1}
\]

Division by \( dt \) in (5.3.1) and a limit process leads to

\[
\frac{db}{dt} = \left(\frac{\partial b}{\partial t}\right)_p + u \left(\frac{\partial b}{\partial x}\right)_p + v \left(\frac{\partial b}{\partial y}\right)_p + \omega \left(\frac{\partial b}{\partial p}\right)_p \tag{5.3.2}
\]

where the subscript \( p \) indicates that the derivatives must be evaluated with the pressure \( p \) held constant, and where we have introduced the notation

\[
\omega = \frac{dp}{dt} \tag{5.3.3}
\]

\( \omega \) in (5.3.3) is the equivalent of the vertical velocity \( w = dz/dt \) in the \( z \)-system. It must be stressed again that the velocity components \( u \) and \( v \) are components of the horizontal velocity vector, and \( dx \) and \( dy \) are horizontal elements. We may get the relationship between the vertical velocity \( w \) and \( \omega \) from the relation

\[
w = \frac{dz}{dt} = \left(\frac{dz}{dt}\right)_p + \vec{V}_h \cdot \vec{V}_p^z + \omega \left(\frac{dV_p}{dp}\right)_p \tag{5.3.4}
\]

or

\[
w = \left(\frac{dz}{dt}\right)_p + \vec{V}_h \cdot \vec{V}_p^z - \frac{a}{g} \omega \tag{5.3.5}
\]

where (5.3.4) is obtained using (5.3.2) with \( b = z \) and (5.3.5) appears when the hydrostatic relation is introduced.

5.4 Transformation Relations

In this section we shall derive the formulae which are necessary to convert from the \( z \)-system to the \( p \)-system. We start by noting that the hydrostatic equation is

\[
\frac{dz}{dp} = -\frac{a}{g} \tag{5.4.1}
\]
which is obtained because of the unique, monotonic relation between \( p \) and \( z \). (5.4.1) is often written in the form

\[
\frac{\partial \psi}{\partial p} = -a
\]  

(5.4.2)

where \( \psi = gz \) is the geopotential.

In obtaining the general transformation relations we shall make use of a visual approach.

![Diagram](image)

**Figure 5.1**

In Figure 5.1 we consider an arbitrary point 1. The horizontal surface \( (z = \text{const.}) \) and the isobaric surface \( (p = \text{const.}) \) through the point are shown in the figure. Let 2 be a point on the horizontal surface a distance \( \delta x \) from 1 in the \( x \)-direction. The vertical line through 2 intersects the isobaric surface at point 3. Considering now an arbitrary scalar \( b \) we note that

\[
\left( \frac{\partial b}{\partial x} \right)_z = \lim_{\delta x \to 0} \frac{b_2 - b_1}{\delta x}
\]

(5.4.3)

while

\[
\left( \frac{\partial b}{\partial x} \right)_p = \lim_{\delta x \to 0} \frac{b_2 - b_1}{\delta x}
\]

(5.4.4)
It follows from the figure that

$$
\left( \frac{\partial b}{\partial x} \right)_z = \lim_{\delta x \to 0} \frac{b_2 - b_1}{\delta x} = \lim_{\delta x \to 0} \left( \frac{b_2 - b_1}{\delta x} - \frac{b_2 - b_2}{\delta x} \right)
$$

(5.4.5)

The first term in (5.4.5) is in the limit $\left( \frac{\partial b}{\partial x} \right)_p$ while the second may be written in the form

$$
\lim_{\delta x \to 0} \frac{b_2 - b_2}{\delta x} = \frac{\partial z}{\partial x} \frac{\partial p}{\partial p} \frac{\partial b}{\partial p} \left( \frac{\partial x}{\partial x} \right)_p
$$

(5.4.6)

where the final factor in the last expression is the slope of the isobaric surface relative to the horizontal surface, hence the subscript p. Consequently

$$
\left( \frac{\partial b}{\partial x} \right)_z = \left( \frac{\partial b}{\partial x} \right)_p + g_p \frac{\partial b}{\partial p} \left( \frac{\partial x}{\partial x} \right)_p
$$

(5.4.7)

Expressions analogous to (5.4.7) can immediately be written for the independent variables $y$ and $t$, i.e.

$$
\left( \frac{\partial b}{\partial y} \right)_z = \left( \frac{\partial b}{\partial y} \right)_p + g_p \frac{\partial b}{\partial p} \left( \frac{\partial y}{\partial y} \right)_p
$$

(5.4.8)

and

$$
\left( \frac{\partial b}{\partial t} \right)_z = \left( \frac{\partial b}{\partial t} \right)_p + g_p \frac{\partial b}{\partial p} \left( \frac{\partial t}{\partial t} \right)_p
$$

(5.4.9)

We find furthermore that

$$
\frac{\partial b}{\partial z} = - g_p \frac{\partial b}{\partial p}
$$

(5.4.10)

The relations (5.4.7) to (5.4.10) are sufficient to perform the transformations required. We note that it follows from (5.4.7) and (5.4.8) that

$$
V_2 b = V_p b + g_p \frac{\partial b}{\partial p} V_p z
$$

(5.4.11)

The vector $V_p z$ is a horizontal vector perpendicular to the isolines for height in the isobaric surface. The vector $V_p b$ is also a horizontal vector because it is the ascendent in the horizontal field which appears when the values of $b$ in the isobaric surface are projected on the horizontal surface. Let us take the temperature field as an example. An analysis of the temperature field in a given isobaric
surface leads to the isotherms on the map. $\nabla_T^p$ is the ascendent for the temperature field and is consequently perpendicular to the isotherms.

Let us consider an arbitrary vector $\vec{b} = b_x \vec{i} + b_y \vec{j}$. The divergence is

$$\nabla_z \cdot \vec{b} = \left( \frac{\partial b_x}{\partial x} \right)_z + \left( \frac{\partial b_y}{\partial y} \right)_z$$

(5.4.12)

From (5.4.7) with $b = b_x$ and (5.4.8) with $b = b_y$ we find by addition

$$\nabla_z \cdot \vec{b} = \nabla_p \cdot \vec{b} + \frac{\partial b}{\partial p} \frac{\partial \vec{b}}{\partial p} \cdot \nabla_p z$$

(5.4.13)

5.5 The Equations of Motion in the $p$-system

The equations in the $z$-system are (1.11.11) because the hydrostatic assumption has been made. We now find that

$$\frac{1}{p} \left( \frac{\partial p}{\partial x} \right)_z = g \left( \frac{\partial z}{\partial x} \right)_p$$

(5.5.1)

and

$$\frac{1}{p} \left( \frac{\partial p}{\partial y} \right)_z = g \left( \frac{\partial z}{\partial y} \right)_p$$

(5.5.2)

from (5.4.7) and (5.4.8) by setting $b = p$. The system (1.11.11) then becomes

$$\left( \frac{\partial u}{\partial t} \right)_p + u \left( \frac{\partial u}{\partial x} \right)_p + v \left( \frac{\partial u}{\partial y} \right)_p + \omega \frac{\partial u}{\partial \theta} = - \frac{\partial \Phi}{\partial x} + fv$$

(5.5.3)

$$\left( \frac{\partial v}{\partial t} \right)_p + u \left( \frac{\partial v}{\partial x} \right)_p + v \left( \frac{\partial v}{\partial y} \right)_p + \omega \frac{\partial v}{\partial \theta} = - \frac{\partial \Phi}{\partial y} - fu$$

(5.5.4)

$$\frac{\partial \Phi}{\partial \theta} = -\alpha$$

(5.5.5)

where we have disregarded friction.
The Continuity Equation in the p-system

The continuity equation in the p-system can naturally be obtained using the formulae in 5.4 and starting from the continuity equation in the z-system. However, in order to avoid a lot of algebra we shall use a derivation similar to the one used in section 2.4. We consider now a "box" bounded by a parallelogram with dimensions \( \delta x \) and \( \delta y \) and an upper and a lower isobaric surface with a pressure difference \( \delta p \) (\( \delta p > 0 \)). The box is fixed in \((x,y,p)\)-space, and the air flows through it. We shall again use the principle that the change of mass is equal to the net inflow of mass. The mass in this case

\[
\delta x \cdot \delta y \cdot \frac{1}{g} \delta p
\]  

(5.6.1)

(5.6.1) is obtained by noting that the mass is \( \rho \delta x \delta y \delta z \) (see Figure 5.2), but since \( \rho \delta z = \frac{\delta p}{\delta p} \) we get (5.6.1). We note that

\[
(x_0,y_0,p_0 - \frac{1}{2} \delta p)
\]

\[
(x_0 - \frac{1}{2} \delta x,y_0,p_0)
\]

\[
(x_0,y_0,p_0 + \frac{1}{2} \delta p)
\]

\[
p_0 - \frac{1}{2} \delta p = \text{const.}
\]

\[
p_0 + \frac{1}{2} \delta p = \text{const.}
\]

Figure 5.2

the expression (5.6.1) is a constant because \( \delta x \), \( \delta y \) and \( \delta p \) are constants.

The net mass inflow in the x-direction is

\[
\rho_1 u_1 \delta y \delta z_1 - \rho_2 u_2 \delta y \delta z_2 = u_1 \delta y \frac{\delta p}{g} - u_2 \delta y \frac{\delta p}{g}
\]

(5.6.2)
or using Taylor expansions

\[
\left[ u_0 - \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)_0 \delta x \right] \delta y \frac{\delta P}{\delta g} - \left[ u_0 + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)_0 \right] \delta x \delta y \frac{\delta P}{\delta g}
\]

which reduces to

\[
- \frac{\partial u}{\partial x} \delta x \delta y \frac{\delta P}{\delta g}
\]

(5.6.4)

The contribution from the y-direction is, by analogy

\[
- \frac{\partial v}{\partial y} \delta x \delta y \frac{\delta P}{\delta g}
\]

(5.6.5)

and from the vertical direction

\[
- \frac{\partial w}{\partial p} \delta x \delta y \frac{\delta P}{\delta g}
\]

(5.6.6)

Since the net mass flux must be zero we get

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial p} = 0
\]

(5.6.7)

which is the continuity equation in the p-system. Note, that (5.6.7) is much simpler in form than the corresponding equation (2.4.4) in the z-system. (5.6.7) has the same form as the continuity equation for the incompressible case in the z-system.

5.7 The Thermodynamic Equation in the p-system

The first law of thermodynamics was derived in section 2.3 and is expressed in (2.3.9) or, more conveniently for the present purpose, in (2.5.11). Since the equation contains only individual derivatives it is a straightforward matter to write it in the p-system. We get from (2.5.11)

\[
\frac{1}{\theta} \frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla \theta + \omega \frac{1}{\theta} \frac{\partial \theta}{\partial p} = \frac{1}{c_p} \frac{\partial H}{\partial p}
\]

(5.7.1)

It is often convenient to express (5.7.1) in terms of the specific volume \( \alpha \) because a through the hydrostatic equation is related to the geopotential which is a readily available quantity in standard isobaric analysis.

We get from the definition of potential temperature
\[
\ln \theta = \ln a - \frac{c_v}{c_p} \ln p + \frac{R}{c_p} \ln p_o - \ln R \quad (5.7.2)
\]

It is seen that
\[
\frac{1}{\theta} \left( \frac{\partial \theta}{\partial t} \right)_p = \frac{1}{a} \left( \frac{\partial a}{\partial t} \right)_p; \quad \frac{1}{\theta} \nabla \theta = \frac{1}{a} \nabla a \quad (5.7.3)
\]

We may therefore write (5.7.1) in the form
\[
\left( \frac{\partial a}{\partial t} \right)_p + \nabla \cdot \nabla a + \omega a \frac{\partial \theta}{\partial p} = \frac{R}{c_p} \frac{1}{p} H \quad (5.7.4)
\]

In the first term of (5.7.4) we introduce
\[
a = -g \frac{dz}{dp} \quad (5.7.5)
\]

and we get
\[
g \frac{\partial}{\partial t} \left( \frac{dz}{dp} \right)_p - \nabla \cdot \nabla a - \omega a \frac{\partial \theta}{\partial p} = \frac{R}{c_p} \frac{1}{p} H \quad (5.7.6)
\]

The last equation is now integrated from an arbitrary isobaric surface \(p_2\) to another isobaric surface \(p_1\). We get \((p_2 > p_1)\):
\[
g \frac{\partial (z_2 - z_1)}{\partial t} + \int_{p_1}^{p_2} \nabla \cdot \nabla a dp + \int_{p_1}^{p_2} \omega a \frac{\partial \theta}{\partial p} dp = \frac{R}{c_p} \int_{p_1}^{p_2} \frac{H}{p} dp \quad (5.7.7)
\]

The quantity \(h = z_1 - z_2\) is the thickness of the isobaric layer from \(p_2\) to \(p_1\). \(h\) is proportional to the mean temperature of the layer as seen from section 4.3. We may write
\[
g \frac{\partial h}{\partial t} = - \int_{p_1}^{p_2} \nabla \cdot \nabla dp + \int_{p_1}^{p_2} \left( - \frac{a}{\theta} \frac{\partial \theta}{\partial p} \right) dp + \frac{R}{c_p} \int_{p_1}^{p_2} \frac{H}{p} dp \quad (5.7.8)
\]

which says that the mean temperature changes according to three processes:

(a) Horizontal advection of air with another temperature.
(b) The coupling of the vertical \(p\)-velocity with the static stability \((-\partial \theta / \partial p\)). If the static stability is positive, i.e. \(\partial \theta / \partial p < 0\), as is normally the case, we find that subsidence \((\omega > 0)\) leads to an increase in the mean temperature of the layer.
(c) The direct effect of the heat source in the column.

We may also consider the temperature at an arbitrary point on an isobaric surface. From (5.7.1) and

\[ \ln \theta = \ln T - \frac{R}{c_p} \ln p + \frac{R}{c_p} \ln p_0 \]  

(5.7.9)

we find

\[ \left( \frac{\partial T}{\partial t} \right)_p + \mathbf{v} \cdot \nabla T + \omega \frac{\partial}{\partial p} \frac{T}{\theta} \frac{\partial \theta}{\partial p} = \frac{1}{c_p} R \]  

(5.7.10)

and we may list the three effects also for the change of temperature in a point.
Chapter VI
CIRCULATION, VORTICITY, DIVERGENCE AND DEFORMATION

6.1 Introduction

The equations of motion, derived in Chapter I and considered again in Chapter V, are equations which give the changes per unit time in the velocity vector, or, in other words, the acceleration. These equations are of paramount importance in dynamic meteorology and are used in prediction problems in the most advanced models, the so-called primitive equation models. It is only within the last few years that the prediction models have been based on the primitive equations in some forecast centres. In other centres and in many diagnostic studies the models are based on the vorticity and divergence equations which are derived equations. The purpose of this section is to introduce the concepts of vorticity and divergence and to derive the equations which govern their development.

6.2 Circulation

Let us consider an arbitrary velocity field $\vec{v}$. In the three-dimensional space we consider two arbitrary points $A$ and $B$ connected by an arbitrary curve (see Figure 6.1). The integral (along the curve)

$$ F = \int_{A}^{B} \vec{v} \cdot d\vec{r} $$

(6.2.1)

where $d\vec{r}$ is an infinitesimal vector along the curve, is called the flow from $A$ to $B$. We note that

$$ \vec{v} \cdot d\vec{r} = V_t \, dr $$

(6.2.2)

where $V_t$ is the component of $\vec{v}$ along the direction given by $d\vec{r}$. It should be stressed that the integral $F$ defined in (6.2.1) will depend in general upon the curve from $A$ to $B$ which is considered in the evaluation.
While the name "flow" for the integral (6.2.1) is attached, in a narrow sense, to the integral of the velocity vector, we may naturally consider the integral of any vector field by an analogous definition. We are thus led to consider integrals of the form:

\[ G = \int_{A}^{B} \vec{b} \cdot d\vec{r} \]  

(6.2.3)

where \( \vec{b} \) is an arbitrary vector field.

We may, in particular, consider a case where the vector \( \vec{b} \) is the ascendent of some scalar field \( \beta \), i.e. \( \vec{b} = \nabla \beta \). We have (see 1.7.7)

\[ d\beta = \nabla \beta \cdot d\vec{r} \]  

(6.2.4)

where \( d\beta \) is the change of \( \beta \) from the beginning of the vector \( d\vec{r} \) to the endpoint of the same vector. We have therefore:

\[ G = \int_{A}^{B} \nabla \beta \cdot d\vec{r} = \int_{A}^{B} d\beta = \beta_{B} - \beta_{A} \]  

(6.2.5)

We shall now consider the special case where the curve is a closed curve (see Figure 6.2).
The integral (6.2.1) is called the circulation in this case, or, in other words, the circulation is defined as the flow along a closed curve. This concept may naturally be generalized to apply to any vector field \( \vec{b} \), and we normally talk about the circulation of \( \vec{b} \) as the integral

\[ C = \oint_{\partial \Omega} \vec{b} \cdot d\vec{r} = \oint_{\bigcap \Omega} \vec{b} \cdot d\vec{r} \quad (6.2.6) \]

where the sign \( \oint \) indicates the integration along a closed curve.

The integral (6.2.5) is zero when the curve is closed. We have therefore:

The circulation of an ascendent is zero along any curve.

We may use this result to show that the circulation of any constant vector along any closed curve is zero. The reason is that if \( \vec{b} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z} \), then we may consider \( \vec{b} \) as the ascendent of the scalar \( \beta = b_x x + b_y y + b_z z \).

6.3 Kelvin's Theorem

We consider a given closed curve \( C \), which, we shall assume, consists of the same particles at all times. Such a curve is sometimes called a physical curve. According to the definitions given in the last paragraph we have:

\[ C = \oint_{\partial \Omega} \vec{b} \cdot d\vec{r} \quad (6.3.1) \]

and

\[ C_{\text{acc}} = \oint \frac{d\vec{b}}{dt} \cdot d\vec{r} \quad (6.3.2) \]

We shall now show that

\[ \frac{dC}{dt} = C_{\text{acc}} \quad (6.3.3) \]

(6.3.3) may be expressed in the form:

The acceleration of the circulation is equal to the circulation of the acceleration (Kelvin's theorem).
We note first that

$$\frac{dC}{dt} = \frac{d}{dt} \left( \mathbf{r} \cdot d\mathbf{r} \right) = \int \frac{d}{dt} \left( \mathbf{v} \cdot d\mathbf{r} \right)$$

(6.3.4)

because the integral is the sum of the contributions from the individual elements and therefore independent of time (since the curve always consists of the same particles). We next note that

$$\int \frac{d}{dt} \left( \mathbf{v} \cdot d\mathbf{r} \right) = \int \frac{dv}{dt} \cdot d\mathbf{r} + \int \mathbf{v} \cdot d(\mathbf{r})$$

(6.3.5)

The first integral in (6.3.5) is $C_{acc}$, and we shall now show that the second integral is zero. In order to do this we consider the curve at a time $t = t_0$ (see Figure 6.3).

Figure 6.3

In a short time $dt$, the curve will have moved to a new position (the dashed curve). Let the points $A$ and $B$ move to the positions $A_1$ and $B_1$. We have then:

$$\overrightarrow{AB} = d\mathbf{r}, \quad \overrightarrow{A_1} = \mathbf{v}_A dt, \quad \overrightarrow{BB_1} = \mathbf{v}_B dt, \quad \overrightarrow{A_1B_1} = d\mathbf{r} + \frac{d(d\mathbf{r})}{dt} dt$$

Since the points $A, A_1, B_1$ and $B$ form a closed polygon we have:

$$\overrightarrow{AA_1} + \overrightarrow{A_1B_1} + \overrightarrow{B_1B} + \overrightarrow{BA} = 0$$

(6.3.6)

or

$$\mathbf{v}_A dt + \left[ d\mathbf{r} + \frac{d(d\mathbf{r})}{dt} dt \right] - \mathbf{v}_B dt - d\mathbf{r} = 0$$

(6.3.7)
or
\[
\frac{d}{dt} (\mathbf{r} \cdot \frac{d \mathbf{r}}{dt}) = \mathbf{v} \cdot \mathbf{r} - \mathbf{v} \cdot \mathbf{v} = \mathbf{d} \mathbf{r}^2
\]  
(6.3.8)

From (6.3.8) it follows that the integrand in the second integral in (6.3.5) is:
\[
\mathbf{v} \cdot \frac{d (\mathbf{r}^2)}{dt} - \mathbf{v} \cdot \mathbf{d} \mathbf{r} = \mathbf{d} \left( \frac{1}{2} \mathbf{v}^2 \right)
\]  
(6.3.9)

and it is now obvious that
\[
\int \mathbf{v} \cdot \frac{d (\mathbf{r}^2)}{dt} - \int \mathbf{d} \left( \frac{1}{2} \mathbf{v}^2 \right) = 0
\]  
(6.3.10)

which completes the proof of Kelvin's theorem.

6.4 Bjerknes' Theorem

The circulation theorem of Bjerknes is a combination of Kelvin's theorem and the equations of motion. We recall that
\[
\frac{d}{dt} \mathbf{v} = -\mathbf{v} \times \mathbf{g} - 2 \Omega \times \mathbf{v} + \mathbf{a}
\]  
(6.4.1)

where we have disregarded friction. We have then:
\[
\frac{d}{dt} \mathbf{c} = -\int \mathbf{v} \times \mathbf{p} \cdot d\mathbf{r} - \int \left( \Omega \times \mathbf{v} \right) \cdot d\mathbf{r} + \int \mathbf{g} \cdot d\mathbf{r}
\]  
(6.4.2)

The last integral in (6.4.2) is zero because \( \mathbf{g} \) is the ascendent of the geopotential. The first term in (6.4.2) may be written in the form:
\[
\int \mathbf{v} \times \mathbf{p} \cdot d\mathbf{r} = \int \mathbf{d} \mathbf{p}
\]  
(6.4.3)

we have therefore:
\[
\frac{d}{dt} \mathbf{c} = -\int \mathbf{d} \mathbf{p} - \int \left( \Omega \times \mathbf{v} \right) \cdot d\mathbf{r}
\]  
(6.4.4)

which is the circulation theorem of Bjerknes. In the following paragraphs we shall make an interpretation of each of the two terms in (6.4.4).
The first term in (6.4.6) depends clearly on the relative positions of the scalar fields of specific volume and pressure. We note that the term is zero if the physical curve is located in an isobaric surface, but this is a very special case. In general, we may note that as a point P (see Figure 6.3) runs through the physical curve in space, we may plot the physical state in each point P' on an (c,p)-diagram. All points on the curve in geometrical space will correspond to a closed curve on the (c,p)-diagram (see Figure 6.4).

![Figure 6.4](image)

We have now

$$\int p_{adp} = \int_{S}^{Q} R(adp) + \int_{Q}^{S} L(adp)$$

(6.4.5)

where the first integral in (6.4.5) is the integral from S to Q along the right (R) part of the curve, while the second integral is from Q to S along the left (L) part of the curve. Now, the first integral is the area \((Q'S'S'RQ'Q')\) (see Figure 6.4). For the second integral we have:

$$\int_{S}^{Q} L(adp) = -\int_{Q}^{S} L(adp)$$

(6.4.6)

and the last integral is the area: \((Q'S'S'LQ'Q')\). We have thus

$$\int p_{adp} = (Q'S'S'RQ'Q') - (Q'S'S'LQ'Q')$$

(6.4.7)

or, in other words, the integral \(\int p_{adp}\) is the area enclosed by the curve in the (c,p)-diagram.
Let us now consider the curve in the \((a,p)\)-diagram once more (see Figure 6.5), and let us consider the isolines \(p, p+1, p+2, \ldots\) and \(a, a+1, a+2, \ldots\).

![Graph showing isolines \((a,p)\) and \((p,a)\) with shaded area]

Figure 6.5

They define a square net over the \((a,p)\)-diagram. The area of each little square is unity because the isobaric surfaces and the isosteric surfaces are drawn at unit intervals. We may now consider the area enclosed by the curve in the \((a,p)\)-diagram as the sum of all the unit squares, each of which is called an isosteric-isobaric solenoid, or just a solenoid.

We may therefore say that the integral \(\int_{a}^{p} dp\) is equal to the number of solenoids enclosed by the curve in the \((a,p)\)-diagram.

or

\[ -\int_{a}^{p} dp = N(a,p) \]  \hspace{1cm} (6.4.8)

6.5 The Solenoidal Vector

The next step is to find the number of solenoids per unit area from the meteorological data of specific volume and pressure.
In Figure 6.6 we have a number of isobaric surfaces, \( p, p+1, p+2, \ldots \) and \( \alpha, \alpha+1, \alpha+2, \ldots \) in geometrical space. The area of the solenoid \( ABCD \) is

\[
\text{(Area)} = (AB)(AD) \sin \theta
\]  \hspace{1cm} (6.5.1)

but

\[
AB = \frac{BB'}{\sin \theta}
\]  \hspace{1cm} (6.5.2)

and

\[
AD = \frac{DD'}{\sin \theta}
\]  \hspace{1cm} (6.5.3)

Thus

\[
\text{(Area)} = \frac{(BB')(DD')}{\sin \theta}
\]  \hspace{1cm} (6.5.4)

We have

\[
|\nabla p| = \frac{1}{DD'} ; \quad |\nabla q| = \frac{1}{BB'}
\]  \hspace{1cm} (6.5.5)

and thus

\[
\text{(Area)} = \frac{1}{|\nabla q| |\nabla p| \sin \theta}
\]  \hspace{1cm} (6.5.6)

The number of solenoids in physical space per unit is then

\[
S = \frac{1}{(\text{Area})} = \frac{1}{|\nabla q| |\nabla p| \sin \theta}
\]  \hspace{1cm} (6.5.7)
If we now introduce the convention that the angle \( \theta \) is counted positive from \( v_0 \) to \( -v_p \), we may define a solenoidal vector:

\[
S = v_0 \times (-v_p)
\]  

where the magnitude of \( S \) is equal to the expression (6.5.7). We note further that \( S \) points into the paper in Figure 6.6.

It will be shown that the number of solenoids in the \((a,p)\)-diagram is equal to the area integral in geometrical space of the solenoidal vector. We have

\[
- \oint_{adp} - \oint_{v vp} \cdot d\vec{r} = - \int_{a} \nabla \times (a \nabla v) \cdot d\vec{A} \tag{6.5.9}
\]

where the last result is obtained using Stokes' theorem. It follows then that

\[
- \oint_{adp} - \oint_{a} (v_0 \times v_p) \cdot d\vec{A} = \oint_{a} \nabla \cdot d\vec{A} \tag{6.5.10}
\]

which proves the statement made above.

6.6 The Coriolis Term

We return now to equation (6.4.4) and consider the second term which comes from the Coriolis term in the equation of motion. We may write this term in the form:

\[
-2 \oint (\Omega \times \vec{v}) \cdot d\vec{r} = -2 \oint (\vec{v} \times d\vec{r}) \cdot \vec{\Omega} \tag{6.6.1}
\]

The vector \( \vec{v} \times d\vec{r} \) is equal in magnitude to the area covered per unit time by the particles located on the vector \( d\vec{r} \). The total integral \( \oint \vec{v} \times d\vec{r} \) is therefore equal to the change per unit time of the area enclosed by the physical curve. We have therefore:

\[
-2 (\oint \vec{v} \times d\vec{r}) \cdot \vec{\Omega} = -2 \vec{\Omega} \cdot \frac{d\vec{A}}{dt} \tag{6.6.2}
\]

where \( \vec{A} \) is the area vector. Now, \( \vec{\Omega} \) points along the axis of rotation of the earth, while \( \vec{\Omega} \) is normal to the plane of the physical curve. We have therefore:

\[
-2 \vec{\Omega} \cdot \frac{d\vec{A}}{dt} = -2 \vec{\Omega} \cdot \frac{dA_E}{dt} \tag{6.6.3}
\]

where \( A_E \) is the area of the projection of \( A \) on the equatorial plane (see Figure 6.7)
With the results obtained in this section we may now write (6.4.4) in the form:

\[
\frac{dC}{dt} = N(a,-p) - 2\Omega \frac{dA_E}{dt}
\]  

(6.6.4)

We notice as a special case that the angle \(\psi\) in Figure 6.7 is the co-latitude if the plane of the curve is horizontal. In that case we may write:

\[
\frac{dA_E}{dt} = \frac{d}{dt}(A \sin \varphi)
\]  

(6.6.5)

where \(\varphi\) is the latitude.

6.7 Applications of the Circulation Theorem

We note that the two terms in (6.6.4) are not equally important in all applications. If for example we are interested in finding the initial circulation from a state of rest it is obvious that we may neglect the last term in (6.6.4). In such a case we find that the increase in the circulation is simply equal to the number of solenoids, and that the increase in the circulation has the same direction as a rotation from \(V_0\) to \(-Vp\). In practice it is very often inconvenient to use \(a\) and \(p\). We notice however that using the gas equation the solenoidal vector may be written in the form:

\[
\vec{S} = -V_0 \times Vp = -\frac{R}{p} \nabla T \times Vp
\]  

(6.7.1)

In vertical cross-sections it is often desirable to use temperature and potential temperature. From the definition of potential temperature we get:
As an application where we may neglect the Coriolis effect initially we consider the phenomenon of the sea-breeze. Let us assume that the air at a certain time is in equilibrium over an underlying surface consisting partly of ocean and partly of land (see Figure 6.8). In the equilibrium state the isobaric and isothermic surfaces are both horizontal. During the day we have a stronger heating of the land than of the ocean. The lower layer of the atmosphere over land will be heated and the specific volume will be larger. The isothermic surfaces will tilt as indicated in Figure 6.8. Along the physical curve (dashed line in Figure 6.8) we will have a circulation in the direction from $\mathbf{v}_o$ to $-\mathbf{v}_p$, i.e. from ocean to land in the lower layer and from land to ocean (return current) in the upper layer.

As the circulation increases in intensity we will get a mass transport of cold air below and warm air above. The isobaric surfaces will start to tilt. In addition, we will no longer be able to neglect the Coriolis effect as the average wind speed along the curve increases. If the coast line is oriented from south to north, we will initially have a sea-breeze from the west in the lower layers. However, as the day progresses, we can expect a change to a more north-westerly direction.
During the night we have reversed relations. The lower layers of the atmosphere over land will be cooled more rapidly than over the ocean. The tilt of the isosteric surfaces relative to the isobaric surfaces will be opposite of the tilt indicated in Figure 6.3, and we will get a land-breeze.

We may consider the effects of the Coriolis term by the following examples. Consider first the circulation around a physical curve which coincides with a latitude circle at a certain time. If the curve is displayed towards the equator, we will get an increase in $A_B$ and therefore a decrease in the circulation, or, in other words, a tendency for the creation of an anticyclonic circulation. If the displacement were towards the pole, we would get an increase in cyclonic circulation.

The next example is concerned with a horizontal physical curve of constant area. If such a curve moves towards the pole and remains horizontal, we will get an increase in $A_B$ and therefore a creation of anticyclonic circulation, while the opposite effect (increase in cyclonic circulation) is observed when the curve moves towards the equator.

We note finally that if we have a horizontal physical curve which remains at a given latitude, contraction will create positive (cyclonic) circulation, while expansion leads to creation of negative (anticyclonic) circulation.

6.8

Differential Properties of the Wind Field

Let us consider a point $(x_0, y_0)$ where the velocity components are $u_o$ and $v_o$. In a sufficiently small neighbourhood of the point $(x_0, y_0)$ we may write the velocity field $(u,v)$ in the form:

$$
\begin{align*}
  u &= u_o + \left( \frac{du}{dx} \right)_o \delta x + \left( \frac{du}{dy} \right)_o \delta y \\
  v &= v_o + \left( \frac{dv}{dx} \right)_o \delta x + \left( \frac{dv}{dy} \right)_o \delta y
\end{align*}
$$

(6.8.1)

where all the derivatives are calculated in the point $(x_0, y_0)$ and where $\delta x = x - x_0$ and $\delta y = y - y_0$. 
It is easily seen that $u$ and $v$ in (6.8.1) may also be written in the form:

\[
\begin{align*}
    u &= u_0 + \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \delta x + \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \delta y \\
    &= \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta y + \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \delta y \\
    v &= v_0 + \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \delta x - \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \delta y \\
    &= \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta x + \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \delta x
\end{align*}
\]

(6.8.2)

Introducing the notations:

\[
\begin{align*}
    D &= \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\
    \zeta &= \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\
    F_1 &= \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \\
    F_2 &= \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)
\end{align*}
\]

(6.8.3)

where $D$ is called the divergence, $\zeta$ the vorticity and $F_1$ and $F_2$ the two components of deformation, we may write (6.8.2) in the form:

\[
\begin{align*}
    u &= u_0 + \frac{1}{2} D \delta x - \frac{1}{2} \xi y + \frac{1}{2} F_1 \delta x + \frac{1}{2} F_2 \delta y \\
    v &= v_0 + \frac{1}{2} D \delta y + \frac{1}{2} \xi x - \frac{1}{2} F_1 \delta y + \frac{1}{2} F_2 \delta x
\end{align*}
\]

(6.8.4)

The developments in (6.8.1) - (6.8.4) are of a purely formal nature. However, the quantities introduced in (6.8.3) are of basic importance in the description of the atmosphere. One of the reasons for their importance is that some of these quantities are invariant under a rotation of the co-ordinate system. Such an invariant quantity may be considered as a property of the atmosphere since it will have the same numerical value independent of the orientation of the co-ordinate system.
It can be shown that \( D, \xi \) and \( F_1^2 + F_2^2 \) are invariant under a rotation of the co-ordinate system. The proof follows the same lines for all three quantities. We shall therefore restrict ourselves to the proof of the invariance of \( D \). We then consider two co-ordinate systems \((x,y)\) and \((x_1,y_1)\) where \((x_1,y_1)\) is obtained from \((x,y)\) by a rotation through the angle \( \theta \), where \( \theta \) is arbitrary, but constant.

We have (see Figure 6.9).

\[
\begin{align*}
x_1 &= x \cos \theta + y \sin \theta \\
y_1 &= -x \sin \theta + y \cos \theta
\end{align*}
\]  
\hspace{6cm} (6.8.5)

or

\[
\begin{align*}
x &= x_1 \cos \theta - y_1 \sin \theta \\
y &= x_1 \sin \theta + y_1 \cos \theta
\end{align*}
\]  
\hspace{6cm} (6.8.6)

When (6.8.5) and (6.8.6) are differentiated with respect to time we get

\[
\begin{align*}
u_1 &= u \cos \theta + v \sin \theta \\
v_1 &= -u \sin \theta + v \cos \theta
\end{align*}
\]  
\hspace{6cm} (6.8.7)
and
\[
\begin{align*}
    u &= u_1 \cos \theta - v_1 \sin \theta \\
    v &= u_1 \sin \theta + v_1 \cos \theta
\end{align*}
\]  
(6.8.8)

We have now:
\[
\begin{align*}
    \frac{du_1}{dx_1} + \frac{dv_1}{dy_1} &= \frac{du_1}{dx} \frac{dx}{dx_1} + \frac{du_1}{dy} \frac{dy}{dx_1} + \frac{dv_1}{dx} \frac{dx}{dy_1} + \frac{dv_1}{dy} \frac{dy}{dy_1} \\
    &= \left(\frac{du}{dx} \cos \theta + \frac{dv}{dx} \sin \theta\right) \cos \theta + \left(\frac{du}{dy} \cos \theta + \frac{dv}{dy} \sin \theta\right) \sin \theta \\
    &\quad + \left(-\frac{du}{dx} \sin \theta + \frac{dv}{dx} \cos \theta\right) (-\sin \theta) + \left(\frac{du}{dy} \sin \theta + \frac{dv}{dy} \cos \theta\right) \cos \theta \\
    &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\
\end{align*}
\]  
(6.8.9)

which completes the proof.

6.9 \textbf{Some Simple Flow Patterns (Streamlines)}

A wind field may be characterized by a family of streamlines at any given time. A streamline is defined as a curve where the tangent to the curve coincides with the velocity vector in all points at a given time. The characterization of a wind field by a set of streamlines is very important in the analysis of meteorological observations, especially, in the tropics, but also in many other areas.
It is seen from Figure 6.10 that the equation for the streamline is

\[
\frac{dy}{dx} = \frac{v}{u} \quad (6.9.1)
\]

Some of the most simple flow fields are obtained by a consideration of special cases of (6.8.4), of which we shall consider four

(a) \( D = \text{const.}, \ \xi = F_1 = F_2 = 0. \)

We then have \( u = \frac{1}{2} D x, \ v = \frac{1}{2} D y. \) The differential equation for the streamline is

\[
\frac{dy}{dx} = \frac{v}{x} \quad (6.9.2)
\]

with the solution

\[
y = Cx \quad (6.9.3)
\]

where \( C \) is an arbitrary constant. The streamlines are therefore all straight lines through \((0,0)\). The case depicted in Figure 6.11 corresponds to \( D > 0. \)

![Figure 6.11](image)

(b) \( \xi = \text{const.}, \ D = F_1 = F_2 = 0. \)

We have \( u = -\frac{1}{2} \xi y, \ v = \frac{1}{2} \xi x \) and
\[ \frac{dy}{dx} = -\frac{x}{y} \quad (6.9.4) \]

which upon integration gives:

\[ x^2 + y^2 = c^2 \quad (6.9.5) \]

The streamlines are all concentric circles with the centre at \((0,0)\). If \(\xi > 0\) we have a counter-clockwise rotation, and a clockwise rotation if \(\xi < 0\) (see Figure 6.12).

\[ \begin{array}{c}
\gamma > \zeta \\
C \\
(0,0) \\
\end{array} \]

Figure 6.12

(c) \[ F_1 = \text{const.}, \quad \zeta = D = F_2 = 0. \]

In this case we have \( u = \frac{1}{2} F_1 x, \quad v = -\frac{1}{2} F_1 y \) and

\[ \frac{dy}{dx} = -\frac{y}{x} \quad (6.9.6) \]

which upon integration gives

\[ xy = C \quad (6.9.7) \]

The streamlines are all hyperbolas with the co-ordinate axes as asymptotes (see Figure 6.13).
(d) $F_2 = \text{const.}, \; \zeta = D = F_1 = 0$.

The velocity components are $u = \frac{1}{2} F_2 y$, $v = \frac{1}{2} F_2 x$, and the equation for the streamline is:

$$\frac{dy}{dx} = \frac{x}{y}$$  \hspace{1cm} (6.9.8)

which gives

$$x^2 - y^2 = \pm c^2$$  \hspace{1cm} (6.9.9)

or all hyperbolas with the lines $x = y$ and $x = -y$ as asymptotes. It is seen that the flow fields in the last two cases (3 and 4) are of the same kind, and both of them are called deformation fields.

The implication of (6.8.4) together with the four simple streamline patterns considered above is that an arbitrary flow-field can be considered as a linear combination of the four fields given above, or in other words, as a sum of fields of divergence, rotation and deformation.

6.10 Interpretation of Vorticity, Divergence and Deformation

In order to obtain a better appreciation of the concepts introduced in
section 6.8 it is useful to consider a further interpretation. The concept of vorticity is related to the instantaneous rate of rotation or the circulation of the atmosphere. To see this we shall consider the circulation around a small rectangular area of the dimensions $\delta x$ and $\delta y$ (see Figure 6.14).

![Diagram of circulation around a small rectangular area]

Figure 6.14

The circulation around this infinitesimal area is

$$
\delta C = \frac{1}{2} \left[ u + u + \frac{du}{\delta x} \delta x \right] \delta x \\
+ \frac{1}{2} \left[ v + \frac{dv}{\delta x} \delta x + v + \frac{dv}{\delta y} \delta y \right] \delta y \\
- \frac{1}{2} \left[ u + \frac{du}{\delta y} \delta y + u + \frac{du}{\delta x} \delta x \right] \delta x \\
- \frac{1}{2} \left[ v + v + \frac{dv}{\delta x} \delta x \right] \delta y
$$

(6.10.1)

or

$$
\delta C = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta x \delta y
$$

(6.10.2)

which shows that the vorticity $\zeta = \partial v / \partial x - \partial u / \partial y$ is equal to the circulation around a unit area, since (6.10.2) can be written in the form:

$$
\zeta = \frac{\delta C}{\delta A}
$$

(6.10.3)

where $\delta A = \delta x \delta y$. 
The expressions (6.10.2) and (6.10.3) are special cases of the general theorem of Stokes which says that the area integral of the vorticity is equal to the circulation around the curve enclosing the area. Mathematically we may express Stokes' theorem in the form:

$$C = \oint_C \mathbf{v} \cdot d\mathbf{r} = \iint_S \mathbf{\omega} \cdot dA \quad (6.10.4)$$

We may make Stokes' theorem plausible (without providing a rigorous proof) by considering Figure 6.15, where the area inside the closed curve has been subdivided into small rectangular areas. For each of these areas we have

$$\delta C = \xi dA \quad (6.10.5)$$

![Figure 6.15](image)

When we sum all the small areas we will in the limit obtain the area integral from the right-hand side of (6.10.5). It is seen from Figure 6.15 that the sum from the left-hand side will be the circulation along the boundary curve because the contributions from the interior line segments vanish. (Each of them appear twice but with opposite sign). It is thus seen that (6.10.4) follows from (6.10.5) by summation over all the small areas.

We may further illustrate the concept of vorticity by considering the circulation around a circle of radius \(r\) in a velocity field corresponding to a rotation with a constant angular velocity \(\omega\). We get:

$$C = \int_0^{2\pi} r \omega d\theta = \int_0^{2\pi} r^2 \omega^2 = 2\pi r^2 \omega \quad (6.10.6)$$
The area of the circle is \( \pi r^2 \), and the circulation per unit area is \( 2\omega \). We find therefore that the vorticity of the motion of solid rotation is equal to twice the angular velocity.

We may therefore interpret the vorticity of a given motion in the following way. Suppose that a small area element suddenly solidified, but otherwise followed the motion of the fluid in contact with it. The vorticity of the area element would then be two times the observed angular velocity of the solid element.

The concept of divergence may be illustrated in a way analogous to that employed for vorticity above. Let us again consider Figure 6.14. The total outflow through the boundary of the rectangular area is

\[
\delta G = -\frac{1}{2} \left[ v + v + \frac{\partial v}{\partial x} \delta x \right] \delta x \\
+ \frac{1}{2} \left[ u + \frac{\partial u}{\partial x} \delta x + u + \frac{\partial u}{\partial y} \delta y \right] \delta y \\
+ \frac{1}{2} \left[ v + \frac{\partial v}{\partial y} \delta y + v + \frac{\partial v}{\partial x} \delta x \right] \delta x \\
- \frac{1}{2} \left[ u + u + \frac{\partial u}{\partial y} \delta y \right] \delta y
\]

(6.10.7)

or

\[
\delta G = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \delta x \delta y
\]

(6.10.8)

which says that the divergence \( D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \) may be considered as the outflow per unit area, because (6.10.8) may be written in the form

\[
D = \frac{\delta G}{\delta A}
\]

(6.10.9)

(6.10.8) may be generalized to an area of finite size leading to Gauss' theorem which may be written in the form:

\[
G = \oint n \, dl = \iint \delta dA
\]

(6.10.10)

where the line integral expresses the total outflow \( (n, \text{ is the normal component, } dl \text{ an element of the curve}) \) while the right-hand side is the area integral of the divergence.
For each of the small elements in Figure 6.16 we find that

\[ \delta G = D \delta A \]  \hspace{1cm} (6.10.11)

Summing over all the small elements we find that the limiting value of the right-hand side will be the area integral of the divergence. The summation of the left-hand side leads to the total outflow through the curve, because the contributions from the interior elements cancel. In the limit we get (6.10.10).

In the considerations in Figure 6.14, and Figure 6.16 we have considered the outflow through a curve fixed in space. We may also follow a fluid element in the motion and thereby get different interpretations. Let us for example consider the rate of change with respect to time of the area

\[ \delta A = \delta x \delta y \]  \hspace{1cm} (6.10.12)

We get

\[ \frac{1}{\delta A} \frac{d(\delta A)}{dt} = \frac{1}{\delta x} \frac{d(\delta x)}{dt} + \frac{1}{\delta y} \frac{d(\delta y)}{dt} \]  \hspace{1cm} (6.10.13)

The meaning of the first term on the right-hand side of (6.10.13) may be obtained from Figure 6.17.
At \( t=t_0 \) we have the element \( \delta x \) in the position \( AB \). A short time \( dt \) later the position is \( A'B' \), and the projection on the \( x \)-axis is \( A''B'' \). We have now:

\[
\frac{1}{\delta x} \frac{d(\delta x)}{dt} = \lim_{\delta t \to 0} \frac{1}{\delta x} \frac{(\delta x') - \delta x}{\delta t} = \lim_{\delta t \to 0} \frac{1}{\delta x} \frac{(x_B'' - x_A'') - (x_B' - x_A')}{\delta t} - \lim_{\delta t \to 0} \frac{1}{\delta x} \frac{(u_B - u_A)}{\delta t} - \frac{\partial v}{\partial x} \tag{6.10.14}
\]

By analogy we find that

\[
\frac{1}{\delta y} \frac{d(\delta y)}{dt} = \frac{\partial v}{\partial y} \tag{6.10.15}
\]

and (6.10.13) may therefore be written in the form:

\[
\frac{1}{\delta A} \frac{d(\delta A)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = D \tag{6.10.16}
\]

or, the divergence is equal to the change in the area of the fluid element per unit time and per unit area.

Using the same technique as above we may obtain an interpretation of the components of the deformation. Consider for example the rate of change per unit time of the ratio of the elements \( \delta x \) and \( \delta y \). We find:

\[
\frac{1}{\delta x/\delta y} \frac{d}{dt} \left( \frac{\delta x}{\delta y} \right) = \frac{1}{\delta x} \frac{d(\delta x)}{dt} - \frac{1}{\delta y} \frac{d(\delta y)}{dt} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = P_l \tag{6.10.17}
\]
which shows that the deformation component \( F_1 \) is equal to the stretching in the \( x \)-direction. For example, if \( \delta x = \delta y \) initially we find that \( \delta x/\delta y > 1 \) after a short time if \( F_1 > 0 \).

6.11 Vorticity Equation

We shall in this section derive an equation for the rate of change of the vorticity \( \zeta \) as defined in the preceding sections. In the present treatment we have restricted our attention to the vorticity \( \zeta = \partial v/\partial x - \partial u/\partial y \), which depends entirely on the horizontal components \( u \) and \( v \), but it should be mentioned that it is possible to define a general three-dimensional vorticity vector of which \( \zeta \) is the vertical component. In order to derive the vorticity equation we shall start from the equation of motion assuming that the hydrostatic relation holds. It is the latter assumption which makes it natural to consider the vorticity component which depends upon the horizontal wind components only, because the vertical velocity is no longer a free variable (see Richardson's equation).

It is naturally possible to derive the vorticity equation in any of the co-ordinate systems considered in the preceding chapter. However, in order to keep the derivation relatively simple, we shall restrict ourselves to the case of the local Cartesian co-ordinate system, with pressure at the vertical co-ordinate.

The equations of motion in this system are:

\[
\begin{align*}
\frac{du}{dt} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} &= - \frac{\partial F_x}{\partial x} + f v + F_x \\
\frac{dv}{dt} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p} &= - \frac{\partial F_y}{\partial y} - f u + F_y
\end{align*}
\] (6.11.1)

In order to obtain an equation involving \( \zeta = \partial v/\partial x - \partial u/\partial y \) we shall differentiate the first equation with respect to \( y \) and the second with respect to \( x \) and subtract the resulting first equation from the resulting second equation. We get:

\[
\begin{align*}
\frac{d\zeta}{dt} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + \omega \frac{\partial \zeta}{\partial p} &+ \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} \right) + \left( \frac{\partial \omega}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial \omega}{\partial y} \frac{\partial u}{\partial x} \right) = \\
-f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - u \frac{\partial f}{\partial x} - v \frac{\partial f}{\partial y} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)
\end{align*}
\] (6.11.3)
Equation (6.11.3) can be reduced by noting that
\[
\left( \frac{du}{dx} \frac{dv}{dy} - \frac{du}{dy} \frac{dv}{dx} + \frac{dv}{dx} \frac{dv}{dy} - \frac{dv}{dy} \frac{du}{dy} \right) = \zeta D \tag{6.11.4}
\]
When we further note that \( \delta f/\delta t = 0 \) and \( \delta f/\delta p = 0 \) we may write (6.11.3) in the form:
\[
\frac{\partial (\zeta + f)}{\partial t} + u \frac{\partial (\zeta + f)}{\partial x} + v \frac{\partial (\zeta + f)}{\partial y} + \omega \frac{\partial (\zeta + f)}{\partial \rho} = \frac{3}{2} \frac{d \rho}{d x} \left( \frac{du}{dx} \frac{dv}{dy} - \frac{dv}{dy} \frac{du}{dx} \right) + \left( \frac{d \rho}{d x} - \frac{d \rho}{d y} \right) - (\zeta + f) D + \left( \frac{du}{dp} \frac{d \omega}{d y} - \frac{dv}{dp} \frac{d \omega}{d x} \right) + \left( \frac{d \rho}{d x} - \frac{d \rho}{d y} \right) \tag{6.11.5}
\]
We introduce the notation
\[
\eta = \zeta + f \tag{6.11.6}
\]
and note that
\[
\frac{du}{dp} \frac{dw}{dy} - \frac{dv}{dp} \frac{dw}{dx} = \mathbf{k} \cdot \left( \frac{dv}{dp} \times \mathbf{v} \omega \right) \tag{6.11.7}
\]
and
\[
\frac{d \rho}{d x} - \frac{d \rho}{d y} = \mathbf{v} \times \mathbf{F} \tag{6.11.8}
\]
Furthermore, when we note that the left-hand side of (6.11.5) is the individual derivative of \( \eta \), we may write (6.11.5) in the form:
\[
\frac{d \eta}{dt} = - \eta D + \mathbf{k} \cdot \left( \frac{dv}{dp} \times \mathbf{v} \omega \right) + \mathbf{k} \cdot \left( \mathbf{v} \times \mathbf{F} \right) \tag{6.11.9}
\]
which we shall call the vorticity equation.

The quantity \( \eta = \zeta + f \) is called the absolute vorticity, because it is the vorticity of the absolute wind.
\[
\mathbf{v}_{a} = \mathbf{v}_{r} + \mathbf{v}_{E} \tag{6.11.10}
\]

The vertical component of the vorticity of \( \mathbf{v}_{r} \) is \( \zeta \) as shown above. In order to find the vertical vorticity component of the earth's velocity \( \mathbf{v}_{E} \) we recall that the motion of the earth is a solid rotation with an angular velocity \( \Omega \). The vorticity of such a motion (see section 6.10) is \( 2\Omega \). The component of \( 2\Omega \)
along the local vertical direction is (see Figure 6.18) $2\Omega \sin \varphi = f$.

![Diagram showing $2\Omega \sin \varphi$ and $\Omega$](image)

**Figure 6.18**

It is thus seen that $\zeta + f$ is the vertical component of the vorticity of the absolute wind $\vec{v}_a$, or, in short, the absolute vorticity.

The terms on the right-hand side of (6.11.9) are called the divergence term, the twisting term and the friction term, respectively. In discussing these terms it should first be noted that $\eta$ is in general a positive number in middle and high latitudes, when we consider motion on a relatively large scale. The value of $f$ at $45^\circ$N is thus approximately $10^{-4}$ s$^{-1}$, while $\zeta$ is normally of the order of magnitude $10^{-5}$ s$^{-1}$. The absolute vorticity can only be negative in strong anticyclonic flow, but even here we find that in general $|\zeta| < f$. It is therefore seen that in a point of divergence ($D > 0$) there will be a tendency to produce anticyclonic vorticity ($d\eta/dt < 0$), while convergence ($D < 0$) will tend to create positive vorticity ($d\eta/dt > 0$).

It is more difficult to see the effect of the twisting term. However, we can illustrate this effect by an example. Consider a flow which is everywhere from the west ($u > 0$, $v = 0$) but increasing with height ($\partial u/\partial p < 0$). Let us further assume that $u$ is a function of $p$ only at a given time $t = t_0$. This situation is illustrated in Figure 6.19. We note that the relative vorticity in point O on plane II is zero.
Let us now consider the effect of the vertical velocity $\omega$. Suppose that $\omega_A$ on plane II is negative, while $\omega_B$ is positive. The effect of the twisting term is

$$\frac{du}{dp} \frac{d\omega}{dy} < 0$$

because $\frac{\partial u}{\partial p} < 0$ and $\frac{\partial \omega}{\partial y} > 0$. From the vorticity equation we then find a tendency $\frac{\partial \zeta}{\partial t} < 0$, or a tendency to create anticyclonic vorticity. We can see this kinematically in the following way. Because $\omega_A < 0$, there is upward motion in A. The upward motion will carry lower windspeed up from below (plane III). On the other hand, $\omega_B > 0$. This means sinking motion in B, and larger windspeeds will be carried down to plane II from plane I. After a short period of time we will have velocities in A and B, where $u_A < u_B$. It follows therefore that

$$\zeta_o = - \frac{du}{\partial y} - \frac{1}{D} (u_B - u_A) < 0.$$ 

As illustrated by the example above we will in general have a change of vorticity in a point due to the twisting effect because the vertical velocity will carry momentum from the pressure surfaces above or below the given surface. Due to this effect of vertical advection in the neighbourhood of a given point we get changes in the horizontal wind shear, and therefore changes in the vorticity.
As a very special, but still important example we see that (6.11.9)
reduces to the equation

\[
\frac{d\eta}{dt} = 0
\]

(6.11.11)
in non-divergent flow without friction. We may see this by noting that \(\omega = 0\) everywhere if \(\mathbf{v} \cdot \nabla = D = 0\), since it follows from the continuity equation that

\[
\frac{\partial \omega}{\partial p} = 0
\]

everywhere, which implies that \(\omega = \text{const}\). However, since \(\omega = 0\) at \(p = 0\) it follows that \(\omega = 0\) everywhere. (6.11.11) says that the absolute vorticity is conserved for a particle moving in a non-divergent, non-viscous flow. This equation has played a major role in the analysis of atmospheric waves because it applies to atmospheric flow in the first approximation. The major reason for this is that the vertical velocity is small compared to the horizontal velocity, and that the divergence is small compared with the vorticity in the atmosphere.

We may also write (6.11.11) in the form:

\[
\zeta + f = \zeta_0 + f_0
\]

(6.11.12)

where \(\zeta_0\) and \(f_0\) are values of the relative vorticity and the Coriolis parameter at some point. If the particle moves to the north \((f > f_0)\) from this position we must expect a decrease in \(\zeta\) from (6.11.12) while \(\zeta\) will increase if the particle moves to the south \((f < f_0)\). We shall return later to a closer analysis of (6.11.11).

As has already been indicated above, not all terms in (6.11.9) are of equal importance. Investigations using partly atmospheric data and partly order of magnitude estimates have given the following results for the vorticity equation. In the equation

\[
\frac{d\zeta}{dt} + u \frac{d(\zeta + f)}{dx} + v \frac{d(\zeta + f)}{dy} + \omega \frac{d\zeta}{dp} = - fD - \zeta D + \mathbf{F} \cdot \left( \frac{\nabla}{\partial p} \times \nabla \right)
\]

(1) (1) (1) (3) (2) (3) (3)

(6.11.13)

we have marked the terms of the greatest magnitude by (1), the terms of smaller importance by (2) and the terms of the least importance by (3). It is thus seen that (6.11.11) contains the terms of the first order of magnitude. An equation containing also the term of second order of magnitude will be

\[
\frac{d\zeta}{dt} + u \frac{d(\zeta + f)}{dx} + v \frac{d(\zeta + f)}{dy} = - fD = f \frac{d\omega}{dp}
\]

(6.11.14)
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This equation is very much used in analysis of large-scale flow because it includes the major effect of the horizontal divergence, or, equivalently, the vertical velocity.

6.12  The Divergence Equation

The divergence equation is derived from the equations of motion (6.11.1) and (6.11.2) by differentiating the first equation with respect to \( x \), the second with respect to \( y \) and adding the resulting equations. Through this process we get

\[
\frac{dD}{dt} + u \frac{dD}{dx} + v \frac{dD}{dy} + \omega \frac{dD}{dp} + \left( \frac{du}{dx} \right)^2 + 2 \frac{du}{dy} \frac{dv}{dx} + \left( \frac{dv}{dy} \right)^2 \\
\frac{d\omega}{dx} \frac{du}{dp} + \frac{d\omega}{dy} \frac{dv}{dp} = -v^2 + f \zeta + v \frac{df}{dx} - u \frac{df}{dy} + v \cdot \overrightarrow{F} \tag{6.12.1}
\]

(6.12.1) may be written in a somewhat shorter form by noting that the first four terms are nothing but \( dD/dt \). For the next three terms we may use the identity:

\[
\left( \frac{du}{dx} \right)^2 + 2 \frac{du}{dy} \frac{dv}{dx} + \left( \frac{dv}{dy} \right)^2 = D^2 + 2J(v,u) \tag{6.12.2}
\]

where \( J(v,u) \) is a short hand notation for the differential expression:

\[
J(v,u) = \frac{dv}{dx} \frac{du}{dy} - \frac{dv}{dy} \frac{du}{dx} \tag{6.12.3}
\]

\( J(v,u) \) is called the Jacobian of \( v \) and \( u \).

The remaining two terms on the left side of (6.12.1) may be written:

\[
\frac{d\omega}{dx} \frac{du}{dp} + \frac{d\omega}{dy} \frac{dv}{dp} = \nabla \cdot \overrightarrow{\omega} \tag{6.12.4}
\]

The divergence equation may now be written in the form:

\[
\frac{dD}{dt} + D^2 + 2J(v,u) + \nabla \cdot \overrightarrow{\omega} + \frac{d\overrightarrow{F}}{dp} = -v^2 + f \zeta + v \frac{df}{dx} - u \frac{df}{dy} + v \cdot \overrightarrow{F} \tag{6.12.5}
\]

(3) (3) (2) (3) (1)

In (6.12.5) we have also introduced the order of magnitudes of the various terms in the divergence equation based on the results of synoptic investigations and the scale analysis. If we include only the first order terms or only the first and
second order terms we get an equation containing no time derivative. Such equations are of diagnostic value and express a balance which must be maintained. Any of the two equations is called a balance equation, and we shall consider them in greater detail later.

6.13 Helmholtz' Theorem

In the analysis of atmospheric flow it is often very useful to consider the vorticity equation and the divergence equation instead of the two equations of motion. Replacing \( u \) and \( v \) we consider \( \zeta \) and \( D \). It is however obvious from (6.11.9) and (6.12.5) that \( u \) and \( v \) still appear in the vorticity and divergence equations. It would therefore be an advantage if we could express the two wind components, the vorticity and the divergence in two scalar quantities. If a solution to this problem can be found we may consider that the vorticity and divergence equations replace the two equations of motion in the true sense of the word.

The problem outlined above was solved by Helmholtz. He showed that an arbitrary wind field \( \vec{v} \) may be written as a sum of two windfields \( \vec{v}_{ND} \) and \( \vec{v}_D \), i.e.

\[
\vec{v} = \vec{v}_{ND} + \vec{v}_D
\]

such that \( \vec{v}_{ND} \) is a non-divergent field and \( \vec{v}_D \) is an irrotational field.

We shall here restrict our attention to two-dimensional, horizontal wind fields. We introduce a stream function, such as

\[
\vec{v}_{ND} = \vec{k} \times \nabla \psi
\]  
(6.13.1)

i.e. \( u_{ND} = -\frac{\partial \psi}{\partial y} \), \( v_{ND} = +\frac{\partial \psi}{\partial x} \). Using (6.13.1) we find that

\[
\zeta_{ND} = \frac{\partial v_{ND}}{\partial x} - \frac{\partial u_{ND}}{\partial y} = \nabla^2 \psi
\]  
(6.13.2)

and

\[
\nabla \cdot \vec{v}_{ND} = \frac{\partial u_{ND}}{\partial x} + \frac{\partial v_{ND}}{\partial y} = 0
\]  
(6.13.3)

The next step is the introduction of a velocity potential such that

\[
\vec{v}_D = \nabla \chi
\]  
(6.13.4)
i.e. \( u_D = \partial x / \partial x \), \( v_D = \partial x / \partial y \). Using (6.13.4) we find that

\[
\xi_D = \frac{\partial v_D}{\partial x} - \frac{\partial u_D}{\partial y} = 0
\]  

(6.13.5)

and

\[
v \cdot \vec{\nabla} D = \frac{\partial u_D}{\partial x} + \frac{\partial v_D}{\partial y} = v^2 x
\]  

(6.13.6)

It is seen from the developments above that

\[
v^2 \psi = \zeta_{ND} = \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
\]  

(6.13.7)

because \( \zeta_D = 0 \) according to (6.13.5). We find also that

\[
v^2 x = v \cdot \vec{\nabla} D = v \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}
\]  

(6.13.8)

because \( v \cdot \vec{\nabla}_{ND} = 0 \) according to (6.13.3).

It is seen from (6.13.7) and (6.13.8) that the stream function \( \psi \) can be found by solving (6.13.7), because the right-hand side is expressed in the known wind field, while the velocity potential can be found by solving (6.13.8), where the right-hand side is expressed as the divergence of the original wind field.

The actual solution of the two equations both of which are Poisson equations goes beyond the scope of this treatment, but each of them can be solved, and it is also possible to show that the partitioning is unique apart from a constant vector.

6.14 Trajectories

The trajectory of a particle is defined as the curve occupied by the particle in the course of time. Sometimes, the trajectory is also called the path of the particle. For each particle in a flow field there exists a trajectory, and we may therefore talk about the family of trajectories for a given collection of particles. The velocity of the particle is at any time directed along the tangent to the trajectory.

The differential equation for the trajectory of a particle is:

\[
\frac{d\vec{R}}{dt} = \vec{v}
\]  

(6.14.1)
where \( \mathbf{r} \) is the radius vector, and \( \mathbf{v} \) is the velocity. The object is to find the variation of the position vector with time when \( \mathbf{v} \) is known. The trajectory is the curve described in space by the endpoint of the position vector.

The solution of (6.14.1) is in practice a very difficult problem because, in general, we observe the atmosphere at fixed stations. Consequently, we do not know the velocity of specific particles, but only the velocity at specific points. The practical construction of a trajectory consists therefore of an approximate solution to (6.14.1).

The basic equation (6.14.1) is equivalent to the three scalar equations:

\[
\frac{dx}{dt} = u; \quad \frac{dy}{dt} = v; \quad \frac{dz}{dt} = w
\]  \hspace{1cm} (6.14.2)

where \( u, v \) and \( w \) are, in general, functions of the space co-ordinates \( x, y, z \) and the time \( t \). Let us consider for example, two-dimensional horizontal motion \((w = 0)\). If \( u \) and \( v \) are independent of time (stationary motion) then we find from (6.14.2) that

\[
\frac{dy}{dx} = \frac{v}{u}
\]  \hspace{1cm} (6.14.3)

which, since the right-hand side is independent of time, is also the equation for the streamlines. We have thus obtained the result that the streamlines coincide with the trajectories when the velocity is independent of time. In the general case there is a difference between the two families of curves for the simple reason that the streamlines apply at a fixed time while the trajectory is by definition the path of a particle.

All the simple cases of streamlines considered in section 6.9 are also trajectories since the velocity components are independent of time. However, it is naturally necessary to solve the equations (6.14.2) in order to determine the time when the particle is at a certain point of the trajectory. Let us consider for instance the example: \( \xi = \text{const.}, \) \( D = F_1 = F_2 = 0 \) of section 6.9. The streamlines and the trajectories are all concentric circles with the origin as centre. The equations for the trajectory starting from an arbitrary point \((x_0, y_0)\) at time \( t = 0 \) with the velocity \((u_0, v_0)\) where \( u_0 = -\frac{1}{2} \xi y_0, \) \( v_0 = \frac{1}{2} \xi x_0, \) are

\[
\frac{dx}{dt} = -\frac{1}{2} \xi y, \quad \frac{dy}{dt} = \frac{1}{2} \xi x
\]  \hspace{1cm} (6.14.4)
Differentiating the first equation with respect to time and substituting from the second gives

$$\frac{d^2 x}{dt^2} + \frac{1}{4} \xi^2 x = 0 \quad (6.14.5)$$

while the equation for $y = y(t)$ is

$$\frac{d^2 y}{dt^2} + \frac{1}{4} \xi^2 y = 0 \quad (6.14.6)$$

The solutions to (6.14.5) and (6.14.6) are

$$x = A \cos \left(\frac{1}{2} \xi t\right) + B \sin \left(\frac{1}{2} \xi t\right) \quad (6.14.7)$$

and

$$y = C \cos \left(\frac{1}{2} \xi t\right) + D \sin \left(\frac{1}{2} \xi t\right) \quad (6.14.8)$$

The four integration constants $A$, $B$, $C$ and $D$ can be determined from the conditions that the particle at $t = 0$ is in the point $(x_0, y_0)$ with the velocity components $u_0 = -\frac{1}{2} \xi y_0$ and $v_0 = \frac{1}{2} \xi x_0$. We get from the first condition that $A = x_0$ and $C = y_0$. The remaining constants $B$ and $D$ are found from the velocities which are

$$u = -\frac{1}{2} \xi A \sin \left(\frac{1}{2} \xi t\right) + \frac{1}{2} \xi B \cos \left(\frac{1}{2} \xi t\right) \quad (6.14.9)$$

$$v = -\frac{1}{2} \xi C \sin \left(\frac{1}{2} \xi t\right) + \frac{1}{2} \xi D \cos \left(\frac{1}{2} \xi t\right) \quad (6.14.10)$$

Substituting $t = 0$ we find from (6.14.9) that

$$-\frac{1}{2} \xi y_0 = \frac{1}{2} \xi B; \quad B = -y_0 \quad (6.14.11)$$

and from (6.14.10) that

$$\frac{1}{2} \xi x_0 = \frac{1}{2} \xi D; \quad D = x_0 \quad (6.14.12)$$

The equations for the trajectory are then

$$x = x_0 \cos \left(\frac{1}{2} \xi t\right) - y_0 \sin \left(\frac{1}{2} \xi t\right) \quad (6.14.13)$$

$$y = y_0 \cos \left(\frac{1}{2} \xi t\right) + x_0 \sin \left(\frac{1}{2} \xi t\right)$$
We notice from (6.14.13) that the time $T$ to complete one revolution is

$$T = \frac{4\pi}{5}$$  \hspace{1cm} (6.14.14)

Furthermore, we may check our result by eliminating the time $t$ from (6.14.13). We find then that

$$x^2 + y^2 = x_0^2 + y_0^2 - r^2$$ \hspace{1cm} (6.14.15)

and we have verified that the trajectory is a circle.

Our next example is somewhat more general because we shall consider a case where the velocity vector is time dependent. Let us consider the wind field

$$u = U = \text{const.}, \quad v = V \cos k (x - ct)$$ \hspace{1cm} (6.14.16)

where $U$ and $V$ are constants. The meridional component $v$ is a function of $x$ and $t$. For a fixed value of time, say $t = t_0$, we notice that the component $v$ is periodic with a period $L$ determined by

$$k \cdot L = 2\pi$$ \hspace{1cm} (6.14.17)

$L = 2\pi/k$ is called the wavelength. It is customary to call $k$ the wave number, but it should be remembered that $k$ according to (6.14.17) is the number of waves in the distance $2\pi$. It is also seen from (6.14.16) that the meridional velocity in the point $x = x_0$ varies periodically in time with the period $T$ determined by

$$k |c| T = 2\pi$$ \hspace{1cm} (6.14.18)

The quantity $v = k |c|$ is called the frequency. We have

$$v T = 2\pi$$ \hspace{1cm} (6.14.19)

and it is seen that $v$ is the number of periods in the interval $2\pi$.

Introducing (6.14.17) in (6.14.18) we find that

$$|c| = \frac{L}{T}$$ \hspace{1cm} (6.14.20)
c therefore has the dimension of a velocity, and is the so-called wave speed, since it measures the velocity with which the wave shape progresses. We may see this by following a fixed point such as the ridge of the wave in time. We note that the ridge is located at \( x = 0 \) at \( t = 0 \). At the time \( T/4 \) we will have the ridge at the point \( x_R \) where

\[
k(x_R - \frac{cT}{4}) = 0 \tag{6.14.21}
\]

or

\[
x_R = \frac{1}{4} cT = \frac{1}{4} L, \quad \text{if } c > 0 \tag{6.14.22}
\]

The wave shape has therefore moved the distance \( L/4 \) in the time \( T/4 \), or in other words \( L/T \) per unit time.

We shall now find the streamlines corresponding to (6.14.16). Let us select the time \( t = 0 \). We have then

\[
\frac{dy}{dx} = \frac{V}{U} \cos kx \tag{6.14.23}
\]

The streamline passing through the point \( x = x_o \) and \( y = y_o \) is

\[
y = y_o + \frac{V}{kU} (\sin kx - \sin kx_o) \tag{6.14.24}
\]

or

\[
y = \frac{V}{kU} \sin kx + C \tag{6.14.25}
\]

where

\[
C = y_o - \frac{V}{kU} \sin kx_o \tag{6.14.26}
\]

According to (6.14.25) the streamlines are sinusoidal curves with the amplitude:

\[
A_s = \frac{V}{kU} \tag{6.14.27}
\]

We shall next find the trajectory which at \( t = 0 \) starts at the point \((x_o, y_o)\). The equations for the trajectory are

\[
\frac{dx}{dt} = U; \quad \frac{dy}{dt} = V \cos k(x - ct) \tag{6.14.28}
\]

Integrating the first part of (6.14.28) we find that
\[ x = x_0 + Ut \]  

Introducing (6.14.29) in the second part of (6.14.28) we get

\[ dy = V \cos \left( kx_0 + k(U - c)t \right) dt \]  

or, by integration

\[ y = y_0 + \frac{V}{k(U - c)} \left[ \sin(kx_0 + k(U - c)t) - \sin kx_0 \right] \]  

or

\[ y = \frac{V}{k(U - c)} \sin k(x - ct) + D \]  

where

\[ D = y_0 - \frac{V}{k(U - c)} \sin kx_0 \]  

Using (6.14.29) we may naturally also write (6.14.32) in the form

\[ y = \frac{V}{k(U - c)} \sin k \left[ \frac{U - c}{U} x + \frac{c}{U} x_0 \right] + D \]  

Using (6.14.32) and (6.14.34) we find that the amplitude of the trajectory is

\[ A_T = \frac{V}{k(U - c)} , \text{ when } U - c > 0 \]  

and we have from (6.14.27) that

\[ A_T = \frac{U}{(U - c)} A_s \]  

The expressions (6.14.31) - (6.14.36) apply when \( c \neq U \). If \( c = U \) we find from (6.14.30) that

\[ y = y_0 + V \cos kx_0 \cdot t \]  

or

\[ y = \frac{V}{U} (\cos kx_0) \cdot x + E \]  

where

\[ E = y_0 - \frac{V}{U} (\cos kx_0) \cdot x_0 \]  

The trajectories are therefore straight lines in this special case.

It is seen from (6.14.25) that the wavelength of the streamlines is

\[ L_s = \frac{2\pi}{k} \]
(6.14.34) tells us that the wavelength of the trajectory is

\[ L_T = \frac{2\pi U}{k(U-c)}, \text{ when } U-c>0 \]  \hspace{1cm} (6.14.41)

From (6.14.40) and (6.14.41) we find that

\[ L_T = \frac{U}{U-c} L_s \]  \hspace{1cm} (6.14.42)

We notice finally that when \( U-c<0 \), we find from (6.14.34) that

\[ y = \frac{V}{k(c-U)} \sin k\left[ \frac{c-U}{U} x - \frac{c}{U} x_0 \right] + D \]  \hspace{1cm} (6.14.43)

and (6.14.36) is replaced by

\[ A_T = \frac{U}{c-U} A_s \]  \hspace{1cm} (6.14.44)

while (6.14.42) becomes

\[ L_T = \frac{U}{c-U} L_s \]  \hspace{1cm} (6.14.45)

The various cases are collected in the following table.

<table>
<thead>
<tr>
<th>( c )</th>
<th>( \frac{A_T}{A_s} )</th>
<th>( \frac{L_T}{L_s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c&lt;0 )</td>
<td>( \frac{A_T}{A_s} &lt; 1 )</td>
<td>( \frac{L_T}{L_s} &lt; 0 )</td>
</tr>
<tr>
<td>( c=0 )</td>
<td>( \frac{A_T}{A_s} = 1 )</td>
<td>( \frac{L_T}{L_s} = 1 )</td>
</tr>
<tr>
<td>( 0 &lt; c &lt; U )</td>
<td>( \frac{A_T}{A_s} &gt; 1 )</td>
<td>( \frac{L_T}{L_s} &gt; 1 )</td>
</tr>
<tr>
<td>( c=U )</td>
<td>not defined</td>
<td>not defined</td>
</tr>
<tr>
<td>( U &lt; c &lt; 2U )</td>
<td>( \frac{A_T}{A_s} &gt; 1 )</td>
<td>( \frac{L_T}{L_s} &gt; 1 )</td>
</tr>
<tr>
<td>( c=2U )</td>
<td>( \frac{A_T}{A_s} = 1 )</td>
<td>( \frac{L_T}{L_s} = 1 )</td>
</tr>
<tr>
<td>( c&gt;2U )</td>
<td>( \frac{A_T}{A_s} &lt; 1 )</td>
<td>( \frac{L_T}{L_s} &lt; 1 )</td>
</tr>
</tbody>
</table>
6.15 Divergence and Vorticity in Spherical Co-ordinates

For the later application of the meteorological equations to the actual atmosphere it becomes important to have the expressions for the horizontal divergence and the vertical component of the vorticity in spherical co-ordinates. We shall derive these expressions here by the use of the expressions (6.10.11) and (6.10.5) For this purpose we shall consider an infinitesimal area \( dA \) bounded by two meridians corresponding to the longitudes \( \lambda - \frac{1}{2} \delta \lambda \) and \( \lambda + \frac{1}{2} \delta \lambda \), respectively, and two latitude circles corresponding to \( \varphi - \frac{1}{2} \delta \varphi \) and \( \varphi + \frac{1}{2} \delta \varphi \), respectively, (see Figure 6.20).

\[
\begin{align*}
\varphi + \frac{1}{2} \delta \varphi \\
\text{a cos} \varphi \delta \lambda \\
\varphi - \frac{1}{2} \delta \varphi \\
\lambda - \frac{1}{2} \delta \lambda \\
\lambda + \frac{1}{2} \delta \lambda
\end{align*}
\]

**Figure 6.20**

We find that

\[
dA = a^2 \cos \varphi \, d\lambda \, d\varphi
\]  
(6.15.1)

For the export we find

\[
dG = \int \mathbf{v} \cdot d \mathbf{l} = u(\lambda + \frac{1}{2} d\lambda, \varphi) \, ad\varphi + v(\lambda, \varphi + \frac{1}{2} d\varphi) \, a \cos (\varphi + \frac{1}{2} d\varphi) \, d\lambda
\]

\[
- u(\lambda - \frac{1}{2} d\lambda, \varphi) \, ad\varphi - v(\lambda, \varphi - \frac{1}{2} d\varphi) \, a \cos (\varphi - \frac{1}{2} d\varphi) \, d\lambda
\]  
(6.15.2)

Using Taylor expansions and standard trigonometric formulae we find

\[
dG = \frac{\partial u}{\partial \lambda} \, ad\varphi + \frac{\partial v}{\partial \varphi} \cos (\frac{1}{2} d\varphi) \, a \cos \varphi d\lambda d\varphi
\]

\[- 2v \sin (\frac{1}{2} d\varphi) \, a \sin \varphi d\lambda
\]  
(6.15.3)
We apply next (6.10.11) saying that
\[ \nabla_h \cdot \mathbf{v} = \lim_{dA} \frac{dC}{dA} \]  
(6.15.4)

and we obtain
\[ \nabla_h \cdot \mathbf{v} = \frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a} \frac{\partial \varphi}{\partial \varphi} - \frac{\tan \varphi}{a} v \]  
(6.15.5)

where we have made use of the facts that \( \cos \varphi \to 1, \varphi \to 0 \) while \( \sin \varphi \varphi \to 1 \), when \( \varphi \to 0 \).

(6.15.5) may also be written in the more compact form:
\[ \nabla_h \cdot \mathbf{v} = \frac{1}{a \cos \varphi} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial \varphi}{\partial \varphi} \cos \varphi \right) \]  
(6.15.6)

The formula for the vertical component of the vorticity is obtained by calculating the circulation around the area element in Figure 6.20. We get
\[
\begin{align*}
\delta C &= \oint_c \mathbf{v} \cdot d\mathbf{l} \\
&= v(\lambda + \frac{1}{2} d\lambda \varphi) \, d\varphi - u(\lambda, \varphi + \frac{1}{2} d\varphi) \, a \cos (\varphi + \frac{1}{2} d\varphi) \, d\lambda \\
&\quad - v(\lambda - \frac{1}{2} d\lambda, \varphi) \, d\varphi + u(\lambda, \varphi - \frac{1}{2} d\varphi) \, a \cos (\varphi - \frac{1}{2} d\varphi) \, d\lambda
\end{align*}
\]  
(6.15.7)

which after evaluation gives
\[
\begin{align*}
\delta C &= \frac{\partial v}{\partial \lambda} a d\lambda d\varphi - \frac{\partial u}{\partial \varphi} \cos (\frac{1}{2} d\varphi) \, a \cos \varphi \, d\lambda d\varphi \\
&\quad + 2u \sin (\frac{1}{2} d\varphi) a \sin \varphi \, d\lambda
\end{align*}
\]  
(6.15.8)

We apply then (6.10.5) giving
\[ \zeta = \lim_{dA} \frac{\delta C}{dA} \]  
(6.15.9)

and we obtain
\[ \zeta = \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a} \frac{\partial u}{\partial \varphi} + \frac{\tan \varphi}{a} u \]  
(6.15.10)

using exactly the same limiting values as before. The compact form of (6.15.10) is:
\[ \zeta = \frac{1}{a \cos \varphi} \left( \frac{\partial v}{\partial \lambda} - \frac{\partial u}{\partial \varphi} \cos \varphi \right) \]  
(6.15.11)

The expressions (6.15.6) and (6.15.11) are the important formulae of this section.
Chapter VII

RECONSIDERATION OF THE PREDICTION PROBLEM

7.1 Introduction

The prediction problem as formulated by Richardson was considered in section 3.3. The main point demonstrated in that section was that when the hydrostatic assumption is made the vertical velocity \( w = \frac{dz}{dt} \) becomes a quantity which can be determined from the horizontal velocity and the physical variables. We may consider the vertical velocity determined as the solution to Richardson's equation as the vertical velocity which must exist in the atmosphere in order to maintain hydrostatic equilibrium everywhere and at all times.

Under the assumption of hydrostatic equilibrium we have also considered in Chapter V all the equations with pressure as the vertical coordinate. The transformation of the equations through the change of the vertical co-ordinate is purely formal and does not change the physical content of the equations. However, it is important to investigate how one can formulate the prediction problem in the p-system. This will be the subject of the next section.

7.2 The Basic Equations in the p-system

The two equations of motion for the horizontal flow, the hydrostatic equation, the continuity equation and the thermodynamic equation in the p-system are

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} = - \frac{\partial \phi}{\partial x} + f v + F_x
\]  
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p} = - \frac{\partial \phi}{\partial y} - f u + F_y
\]  
\[
\frac{\partial \phi}{\partial p} = - \frac{R}{p} \left( \frac{p}{p_0} \right)^\frac{C_p}{C_v} \theta
\]  
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0
\]  
\[
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + \omega \frac{\partial \theta}{\partial p} = \frac{1}{C_p} \left( \frac{p}{p_0} \right)^{C_p} H
\]
A few words of explanation are necessary to explain the form of these
equations. We have originally obtained the hydrostatic equation in the form
\[
\frac{\partial \phi}{\partial p} = -a
\]  
(7.2.6)
but if we substitute from the gas equation
\[
a = \frac{RT}{P}
\]  
(7.2.7)
and the definition of the potential temperature
\[
T = \theta \left( \frac{P}{P_0} \right)^{\frac{R}{c_p}}
\]  
(7.2.8)
we obtain (7.2.3)

We have furthermore written the thermodynamic equation in terms of the
potential temperature, and the five equations have the five variables, \(u, v, \omega, \phi\) and
\(\theta\). Three of the equations (7.2.1), (7.2.2) and (7.2.5) are prognostic equations
which will be used to find \(u, v\) and \(\theta\) a time step later, while the remaining two
equations, (7.2.3) and (7.2.4), will be used to obtain the values of \(\phi\) and \(\omega\) from
the already computed values of \(u, v\) and \(\theta\) at the new time. It is seen that both \(\phi\)
and \(\omega\) must be obtained from an integration in the vertical direction, and it is thus
necessary to specify boundary conditions.

7.3 Boundary Conditions

The boundary condition on \(\omega\) at the outer limit of the atmosphere is
\[
\omega = 0, \quad p = 0
\]  
(7.3.1)
because the pressure approaches zero at the limit and the pressure change for a
particle can therefore also only be equal to zero.

The boundary condition at the earth's surface is, as mentioned in section
3.2, that the normal component of the velocity vector vanishes. We may therefore
write that
\[
\omega_s = \left( \frac{dp}{dt} \right)_s \left( \frac{\partial p}{\partial t} \right)_s + \vec{v}_s \cdot \nabla p_s
\]  
(7.3.2)
where \(s\) denotes the surface of the earth, \(\vec{v}_s\), the wind component along the surface, and
\(p_s = p_s(x,y,t)\) the surface pressure. Because \(p_s\) is a function of \(x, y\) and \(t\) only we
may write $\vec{v}_{sh} \cdot \nabla P_s$ in (7.3.2) where $\vec{v}_{sh}$ is the horizontal component of $\vec{v}_s$.

From the continuity equation we have

$$\omega_s = -\int \nabla P_s \cdot \nabla \psi$$  \hspace{1cm} (7.3.3)

Equating the expressions (7.3.2) and (7.3.3) we get

$$\frac{\partial P_s}{\partial t} = -\int \nabla P_s \cdot \nabla \psi - \vec{v}_{sh} \cdot \nabla P_s$$  \hspace{1cm} (7.3.4)

which is the correct boundary condition at the surface of the earth.

We consider next the isobaric surface through the point in question, and we have (see section 5.4) that

$$\frac{\partial P_s}{\partial t} = \nabla P_s \left( \frac{\partial \phi}{\partial t} \right)_{s,p} \cdot \nabla P_s = \nabla P_s \cdot \nabla \phi$$  \hspace{1cm} (7.3.5)

and (7.3.4) becomes

$$\left( \frac{\partial \phi}{\partial t} \right)_{s,p} = -\alpha_s \int \nabla P_s \cdot \nabla \psi - \vec{v}_{sh} \cdot \nabla P_s$$  \hspace{1cm} (7.3.6)

which is another form of the lower boundary condition. In using (7.3.6) it is most often assumed that $P_s \approx P_0 = 1000$ mb. In that case $\phi_{s,p}$ becomes the geopotential of the 1000 mb isobaric surface, and

$$\left( \frac{\partial \phi_{s,p}}{\partial t} \right) = -\alpha_s \int \nabla P_0 \cdot \nabla \psi - \vec{v}_0 \cdot \nabla \phi_0$$  \hspace{1cm} (7.3.7)

which is the equation which can be used to calculate the new values of the geopotential at the lower (isobaric) boundary.

7.4 The Prediction Problem

The equations (7.2.1) - (7.2.5) combined with (7.3.7) form the basis for the prediction system using pressure as the vertical co-ordinate.

Equations (7.2.1), (7.2.2), (7.2.5) and (7.3.7) are used to compute new values of $u$, $v$, $\theta$ and $\phi_0$ by a system of finite differences of these equations. We shall not here be interested in the details of the finite difference system except
to note that the atmosphere is divided into elementary volumes by a three-dimensional grid. All partial derivatives in space and time are replaced by finite differences of the values of the variables in the grid points. It is then seen that all terms except the time-derivatives in (7.2.1), (7.2.2), (7.2.5) and (7.3.7) can be calculated from values of \( u, v, \omega, \phi \) and \( \theta \) at a given time. The time derivatives in these equations are therefore known, and we may obtain values of \( u, v, \theta \) and \( \phi \) a short time later. These values will be called the new values.

From the new values of \( \theta \) and \( \phi \), we obtain the new values of \( \phi \) from an integration of (7.2.3) which gives

\[
\phi(p) = \phi_o + \int_{p_0}^p \frac{R}{p} \left( \frac{p}{p_o} \right)^c dp
\]  

(7.4.1)

The integral (7.4.1) is replaced by a finite sum. Since the isobaric surfaces remain the same throughout the time-integration we may calculate the pressure dependent factor in the integrand once and for all.

The new values of \( \omega \) are obtained by integrating (7.2.4). We get

\[
\omega(p) = -\int_{p_0}^p \nabla p \cdot \nabla dp
\]  

(7.4.2)

We have then new values of all the variables and the whole process can be repeated.

It should be noted in (7.4.1) that the formulation of the lower boundary condition in the form of a prognostic equation for \( \phi_0 \) is very important in providing the value of the integration constant in the integration of the hydrostatic equation. Note furthermore that the upper boundary condition \( \omega = 0, p = 0 \) plays a similar role in the integration (7.4.2) of the continuity equation.

The system described here in the \( p \)-system is equivalent to Richardson's prediction system as described in Chapter III. Equation (7.4.2) is the equivalent of Richardson's equation for the vertical velocity, and it is obvious how much simpler the equation for \( \omega \) is as compared with (3.3.5) and (3.3.6) for \( w \).

We note finally that the information necessary to start the integration are the fields of \( u, v, \theta \) and \( \phi \). The requirements are well suited to the present observation system where the radiosonde measures the horizontal wind vector at the various
levels and the temperature as a function of pressure. It is thus easy to compute \( \theta \) from
\[
\theta = T \left( \frac{p}{p_0} \right)^{-\frac{R}{C_p}} \tag{7.4.3}
\]
and \( \varphi_0 \) from the formula (see equation (4.3.4)).
\[
\varphi_0 = \varphi_s + RT_m \ln \left( \frac{p_s}{p_0} \right) \tag{7.4.4}
\]
where \( \varphi_s \) is the geopotential at the surface of the earth.

One of the disadvantages in using height or pressure as the vertical co-ordinate is that the lower boundary of the atmosphere determined by the continental elevations and the ocean surface, differs quite a lot from the co-ordinate surfaces in the two systems. The horizontal surfaces in the \( z \)-system and the isobaric surfaces in the \( p \)-system intersects the lower boundary surface in the region of high elevations. We noticed this difference in going from (7.3.4) to (7.3.7). It is also evident that difficulties will arise when the isobaric surface is partly in the atmosphere and partly "under ground" in the vicinity of a gridpoint. Because of these difficulties it is necessary to take special care in evaluating the pressure force, for example. Co-ordinate systems in which the lower boundary is a co-ordinate surface have been designed. The vertical co-ordinate in such a system is the actual pressure divided by the surface pressure, i.e.
\[
\sigma = \frac{p}{p_s} \tag{7.4.5}
\]

It is seen that \( \sigma = 0 \) at the outer limit of the atmosphere, while \( \sigma = 1 \) at the earth's surface. This co-ordinate system, the \( \sigma \)-system, is used in practical predictions, but the description of the details of the transformation is beyond the scope of this compendium.
Chapter VIII

BALANCED MOTION

8.1 Introduction

A balanced motion is one where the forces acting upon the particle at any time add up to zero. In the most important meteorological applications we count the apparent forces such as the Coriolis force and the centrifugal force as real forces. As a simple example of balanced motion we may mention the rotation of a cylindrical vessel containing water of constant density around a vertical axis through the centre of the fluid (see Figure 8.1).

![Figure 8.1](image)

If the container is put on a turn-table and rotated with a constant angular velocity one will observe that the free surface of the water after a while assumes a shape as indicated in Figure 8.1. Each particle in the fluid describes a circular motion with the centre on the axis of rotation. The real force acting on the particle is the horizontal pressure force, while a centripetal acceleration keeps the particle in the circular trajectory. We may consider this as a balanced motion where the balance is between the pressure force $\vec{F}$ and the centrifugal force $\vec{C}$. The balance can be used to calculate the shape of the free surface. We have:

$$ - \frac{1}{\rho} \frac{d\rho}{dr} - \vec{\Omega} \times (\vec{\Omega} \times r) = 0 $$  \hspace{1cm} (8.1.1)

From the hydrostatic equation assuming that $p = 0$ at the free surface we obtain

$$ p = \rho (H - z) $$  \hspace{1cm} (8.1.2)
where \( H = H(r) \) is the depth of the fluid. We have therefore

\[
\frac{1}{\rho} \frac{dp}{dr} = g \frac{dH}{dr}
\]  

(8.1.3)

and (8.1.1) becomes

\[
\frac{dH}{dr} = \frac{\Omega^2}{g} r
\]  

(8.1.4)

which upon integration gives

\[
H = H_0 + \frac{1}{2} \frac{\Omega^2}{g} r^2
\]  

(8.1.5)

where \( H_0 \) is the depth at \( r = 0 \). It is thus seen that the shape of the free surface is a parabola.

8.2 The Geostrophic Wind

The geostrophic wind is defined as the wind which would exist in the atmosphere if the motion were hydrostatic and horizontal without acceleration and friction.

It is seen that the equations for the horizontal motion in this special case reduce to

\[
0 = -\frac{1}{\rho} \frac{\partial p}{\partial x} + f v_g
\]  

(8.2.1)

\[
0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} + f u_g
\]  

(8.2.2)

as seen from equations (1.11.11). The subscript \( g \) in (8.2.1) and (8.2.2) denotes "geostrophic". It is also seen from the equations above that the case of the geostrophic wind represents a balanced motion where the balance is between the pressure force and the Coriolis force.

The fact that the acceleration is zero in geostrophic motion implies that the motion is along a straight line. A curved trajectory would require a normal acceleration. Since we know that the pressure force is normal to the isobar and pointing from higher to lower pressure, it follows that the Coriolis force is also normal to the isobar, but points from lower to higher pressure, in geostrophic motion. The geostrophic wind vector will be pointed along the isobar (which is a straight line) with the low pressure to the left looking downstream (in the northern hemisphere) because the Coriolis force is perpendicular to the wind vector and points to the right (in the northern hemisphere). These relations are shown in Figure 8.2, where \( \vec{v} \) is the Coriolis force.
It follows from (8.2.1) and (8.2.2) that

\[ u_g = -\frac{1}{\rho_f} \frac{\partial p}{\partial y} \]  
(8.2.3)

\[ v_g = \frac{1}{\rho_f} \frac{\partial p}{\partial x} \]  
(8.2.4)

and it is easy to verify that the geostrophic wind vector may be written in the form

\[ \vec{V}_g = \frac{1}{\rho_f} \vec{k} \times \vec{v}_p \]  
(8.2.5)

where \( \vec{k} \) is a vertical unit vector pointing upwards. From (8.2.5) it follows that the magnitude of \( \vec{V}_g \) is

\[ V_g = \frac{1}{\rho_f} |\vec{v}_p| = \frac{1}{\rho_f} \frac{\partial p}{\partial n} \]  
(8.2.6)

where \( \frac{\partial p}{\partial n} \) is the variation of pressure along a normal to the isobar.

The formulae developed above apply to the \( z \)-system. It is easy to see from (5.5.3) and (5.5.4) that the equations for the geostrophic wind in the \( p \)-system are

\[ 0 = -\frac{\partial \Phi}{\partial x} + f v_g \]  
(8.2.7)

\[ 0 = -\frac{\partial \Phi}{\partial y} - f u_g \]  
(8.2.8)

The relations of the forces to the geostrophic wind expressed by (8.2.7) and (8.2.8) are shown in Figure 8.3.
We find

\[ u_g = -\frac{1}{f} \frac{\partial \phi}{\partial y} \]  \hspace{1cm} (8.2.9)

\[ v_g = \frac{1}{f} \frac{\partial \phi}{\partial x} \]  \hspace{1cm} (8.2.10)

and

\[ v_g = \frac{1}{f} \mathbf{k} \times \mathbf{v}\phi \]  \hspace{1cm} (8.2.11)

from which it follows that

\[ V_g = \frac{1}{f} |\mathbf{v}\phi| = \frac{1}{f} \frac{\partial \phi}{\partial n} \]  \hspace{1cm} (3.2.12)

The fact that the magnitude of the geostrophic wind as computed on an isobaric surface depends on the magnitude of the geopotential gradient and the Coriolis parameter only, was very instrumental in introducing the isobaric surfaces as reference surfaces in the practical analysis of the atmosphere. It is seen that the corresponding formula (8.2.6) for the horizontal surfaces contains also the density which complicates the picture, because a given pressure gradient at a given location will result in very different geostrophic winds at the surface of the earth and at, say, 10 km. On the other hand (8.2.12) shows that the magnitude of the geostrophic wind will be the same at all isobaric levels if the geopotential gradient and the location are the same.

As seen from all the formulae the geostrophic windspeed is proportional to the magnitude of the pressure force and inversely proportional to the magnitude of
the Coriolis parameter. A given spacing of the isohyprs (lines of constant geopotential) will thus give different geostrophic speeds at different latitudes. We find that the geostrophic speed becomes smaller the higher the latitude for the same value of the pressure force.

The concept of the geostrophic wind breaks down at the equator where $f = 0$ and the Coriolis force vanishes.

8.3 Geostrophic Winds and Actual Winds

The geostrophic wind can be calculated entirely from the geopotential field on an isobaric surface and from the knowledge of position which determines the Coriolis parameter. The definition of the geostrophic wind sounds extremely restrictive because it requires horizontal motion without acceleration. When it is further noted that the geostrophic wind requires straight isohyprs on isobaric surfaces (or straight isobars on horizontal surfaces) it seems at first sight that the applicability of this artificial wind to the actual atmosphere with its curved isohyprs, accelerations and frictional forces must be very limited. Quite the contrary is the case. The reason is that the largest terms in the equations for the horizontal flow are the pressure force and the Coriolis force, while the remaining terms such as the acceleration and the frictional force are at least an order of magnitude smaller than the pressure force (and the Coriolis force). The statement made above is based on an estimate of all terms in the equations for the horizontal motion from meteorological data for large-scale synoptic motion, and it is therefore entirely of an empirical nature.

The much more basic question of why the atmospheric large-scale flow is such that it remains in almost geostrophic balance is difficult to answer. No complete answer is known at the present time, and it is beyond the scope of this treatment to describe the various approaches to the problem.

As we would expect, we find the largest systematic deviations from the geostrophic flow in strongly curved flow. Empirical comparisons between the geostrophic wind, computed from analysed weather maps, and the actual wind, as measured by radiosondes, show that the geostrophic wind is much larger than the actual wind in flow which has a strong cyclonic curvature, while it is difficult to find a strongly systematic difference in anticyclonic flow.
We shall in the next section investigate some aspects of balanced flow when curvature is present. From the results of this investigation we shall be able to explain the empirical differences mentioned above.

8.4 The Gradient Wind

The gradient wind is defined as the wind which exists if the trajectory of the particle is circular, and we have a balance between the pressure force, the Coriolis force and the centrifugal force.

It is seen immediately that the gradient wind is a generalization of the geostrophic wind, since that case requires a balance of the pressure force and the Coriolis force. As seen in section 8.2 we must have a straight trajectory coinciding with the isobar if we have strict geostrophic balance. We may therefore say that we consider the effects of a curved trajectory on the windspeed in the case of the gradient wind.

It is convenient to consider the cases of counter-clockwise and clockwise circulations separately. In the first case, see Figure 8.4, we have a Coriolis force \( \mathbf{C} \) and a centrifugal force \( \mathbf{C}_e \) which act in the same direction as the radius vector. In order to have a balance, the pressure force \( \mathbf{P} \) must act in the opposite direction. We have therefore a low pressure in the centre. The numerical value of the pressure force is \( \alpha(dp/dR) \), where \( R \) is the distance from the centre. We may express the pressure force in terms of the geostrophic wind. As seen from (8.2.6) (or equivalently (8.2.12) if we deal with an isobaric surface) we have:
\[
\left| \frac{dP}{dR} \right| = fV_g \\
\text{(8.4.1)}
\]

We shall denote the gradient wind by \( V_G \), and we have
\[
|\vec{C}| = fV_G \\
\text{(8.4.2)}
\]

and
\[
|\vec{C}_g| = \frac{V_G^2}{R} \\
\text{(8.4.3)}
\]

where the last expression is obtained from (1.2.15). According to Figure 8.4 we can find the magnitude of the gradient wind by solving the equation
\[
fV_G + \frac{V_G^2}{R} - fV_g = 0 \\
\text{(8.4.4)}
\]

It turns out that it is most convenient to solve first for \( 1/V_G \). We find
\[
fV_g \left( \frac{1}{V_G} \right)^2 - f \frac{1}{V_G} - \frac{1}{R} = 0 \\
\text{(8.4.5)}
\]
or
\[
\frac{1}{V_G} = \frac{f + \sqrt{f^2 + \frac{4fV_g}{R}}}{2fV_g} \\
\text{(8.4.6)}
\]

and
\[
V_G = \frac{g}{\frac{1}{2} \left( \frac{1}{V_g} + \frac{V_g}{R} \right)} \\
\text{(8.4.7)}
\]

We notice in (8.4.7) that only the plus sign has physical meaning. We may see this by letting \( R \to \infty \). In that case we must require that
\( V_G \to V_g \), and that is the case only when the plus sign is used.

Equation (8.4.7) tells us that \( V_G < V_g \) in all cases because the denominator is larger than 1. The difference between \( V_G \) and \( V_g \) becomes larger the smaller \( R \) is, and the smaller \( f \) is. To illustrate the differences we may consider a geostrophic windspeed of \( V_g = 10 \text{ m s}^{-1} \) and a latitude of \( 45^\circ \text{N} \) (\( f = 10^{-4} \text{ s}^{-1} \)). If \( R = 1000 \text{ km} = 10^6 \text{ m} \) which may be typical of atmospheric flow, we find \( V_G = 9.16 \text{ m s}^{-1} \), and the difference between \( V_G \) and \( V_g \) is small. It is only when \( R \) becomes much smaller, i.e. in flow with strong curvature, that the difference between \( V_G \) and \( V_g \) will be large.

If we assume once again that \( f = 10^{-4} \text{ s}^{-1} \) and \( V_g = 10 \text{ m s}^{-1} \) we may calculate the value of \( R \) necessary to make \( V_G = \frac{1}{2} V_g \). We find from (8.4.7) that a radius of \( R \approx 200 \text{ km} \) is required.
The case of clockwise circulation is illustrated in Figure 8.5 using the same symbols as earlier. Under the assumption of high pressure in the centre as in Figure 8.5 we get the equation

$$fV = \frac{V^2}{R} - fV_G = 0 \quad (8.4.8)$$

in which the first term is the pressure force, the second the centrifugal force and the last term is the Coriolis term. A solution of (8.4.8) using the same technique as with (8.4.4) gives:

$$V_G = \frac{V}{\frac{1}{2} + \frac{1}{4} - \frac{V}{Rf}} \quad (8.4.9)$$

where the minus sign has been removed for the same reason as in (8.4.7). We note immediately that (8.4.9) may be obtained from (8.4.7) by the convention that $R$ is positive for counter-clockwise and negative for clockwise circulation in agreement with the general rules of geometry.

It is seen from (8.4.9) that $V_G > V$ in all cases where $V_G$ exists. However, if the radius of curvature becomes too small for given values of $V$ and $f$, it is seen that the quantity under the radical may become negative, and $V_G$ does not exist. In the limiting case, where

$$\frac{V}{Rf} = \frac{1}{4} \quad (8.4.10)$$

we find

$$V_G = 2V \quad (8.4.11)$$
The maximum gradient wind in the anticyclonic case is therefore twice the geostrophic wind. The radius of curvature to accomplish this is, for $f = 10^{-4} \text{s}^{-1}$ and $V_g = 10 \text{ m s}^{-1}$, equal to 400 km, and thus quite small. (6.4.9) shows that the limiting case is reached, when it becomes impossible to balance the forces. For given values of $f$ and $V_g$, we have a constant pressure force $\vec{F}$. If we let $R$ decrease, $V_g$ will increase as seen from (6.4.9). However, the centrifugal force $(V_g^2/R)$ will increase as $V_g^2$, while the Coriolis force $(fV_g)$ will increase as $V_g$ only. We will therefore inevitably arrive at a situation where a balance of forces is impossible, which means that the particle will leave the circular motion. Conditions which come close to such a situation are found in strong anticyclonic flow in ridges in the upper troposphere.

Since the gradient wind is somewhat more general than the geostrophic wind because it pays attention to curvature effects, we should expect that the agreement between the gradient wind and the actual flow would be somewhat closer than the comparison between the geostrophic wind and the observed wind. This is also the case, especially in cyclonic flow, where the geostrophic wind is an overestimate of the actual wind. The somewhat smaller gradient wind is a better approximation in this case. A similar improvement is not found in the case of anticyclonic flow, presumably because the wind speeds in anticyclonic flow are in general much smaller than in cyclonic flow. We shall, however, return to this question in the next section.

8.5 Motion with Acceleration

We shall consider a flow which is somewhat more general than the geostrophic and the gradient wind flows. In this case we shall allow a deviation in both speed and direction between the actual flow and the geostrophic flow.

![Figure 8.6](image)
The equations of motion may be written in the form

\[ \frac{du}{dt} = f(v - v_g) \]  
(8.5.1)

\[ \frac{dv}{dt} = -f(u-u_g) \]  
(8.5.2)

because the components of the pressure force may be written

\[ -a \frac{\partial p}{\partial x} = -fv_g \]  
(8.5.3)

\[ -a \frac{\partial p}{\partial y} = fu_g \]  
(8.5.4)

From Figure 8.6 we get

\[ u = V \cos \psi; \quad u_g = V \cos \delta \]  
(8.5.5)

\[ v = V \sin \psi; \quad v_g = V \sin \delta \]  
(8.5.6)

It then follows that

\[ \frac{du}{dt} = \frac{dV}{dt} \cos \psi - V \frac{d\psi}{dt} \sin \psi = fV \sin \psi - fV_g \sin \delta \]  
(8.5.7)

\[ \frac{dv}{dt} = \frac{dV}{dt} \sin \psi + V \frac{d\psi}{dt} \cos \psi = fV \cos \psi + fV_g \cos \delta \]  
(8.5.8)

These equations are solved for \( \frac{dV}{dt} \) and \( V \frac{d\psi}{dt} \) and we get

\[ \frac{dV}{dt} = fV_g \sin \epsilon \]  
(8.5.9)

\[ V \frac{d\psi}{dt} = -fV + fV_g \cos \epsilon \]  
(8.5.10)

We notice that the curvature of the trajectory is defined by the relation

\[ K_t = \frac{d\psi}{ds} \]  
(8.5.11)

where \( ds \) is an infinitesimal part of the trajectory. Dividing by \( dt \) we find

\[ \frac{d\psi}{dt} = K_t \frac{V}{R_t} \]  
(8.5.12)
where \( R_t \) is the radius of curvature of the trajectory. We may therefore write

\[
\frac{d\nu}{dt} = f\nu \sin \epsilon \tag{8.5.13}
\]

\[
\frac{\nu^2}{R_t} = -f\nu + f\nu g \cos \epsilon \tag{8.5.14}
\]

We could also have obtained these equations directly by noting that the
tangential acceleration is \( d\nu/dt \), while the normal acceleration is \( \nu^2 R_t \) according to
(1.2.15). The right-hand sides (8.5.13) and (8.5.14) are the sums of the components of the forces in the tangential and normal directions, respectively (see Figure 8.7).

![Figure 8.7](image)

The curve in Figure 8.7 is a part of the trajectory. At point \( A \) we have the velocity \( \nu \) and the geostrophic velocity \( \nu g \). \( \epsilon \) is the angle between the vectors. The pressure force \( \vec{P} \) is perpendicular to \( \nu g \) and \( \vec{P} = f\nu g \). On the other hand, the Coriolis force \( \vec{C} \) is perpendicular to \( \nu \) and directed to the right. The centrifugal force \( \vec{C}_e \) is opposite to the centripetal acceleration.

The only force which has a component in the direction of \( \nu \) is \( \vec{P} \), and the component is \( P \cos \left( \frac{\nu}{2} \pi - \epsilon \right) = f\nu \sin \epsilon \), which gives us (8.5.13). The other equation (8.5.14) is obtained by a balance of the components in the direction of the normal.

We may solve (8.5.14) in the same way as the gradient wind equation by first solving for \( 1/\nu \). The expression for \( \nu \) becomes:
\[ V = \frac{V_g \cos \epsilon}{\frac{1}{2} \sqrt{\frac{1}{4} + \frac{V_g \cos \epsilon}{fR_t}}} \]  

(8.5.15)

(8.5.13) and (8.5.15) are more general than the gradient wind equation (3.4.7) (or (8.4.9)), since accelerations are permitted in the equations of the present section, and because a deviation between the actual and the geostrophic wind is permitted. However, we observed that if we restrict ourselves to the case of no tangential acceleration, i.e. \( \frac{dV}{dt} = 0 \), we find from (8.5.13) that \( \epsilon = 0 \), and (8.5.15) will in this case reduce to the gradient wind expression (3.4.7). The gradient wind may therefore also be defined as the wind which would exist if there were no tangential acceleration.

Observations show that a deviation \( \epsilon \) often exists between the actual and the geostrophic wind. When we therefore reduce (8.5.15) to (3.4.7) by setting \( \epsilon = 0 \), we make an approximation. In the case of \( R_t > 0 \) (cycloonic flow) we make both numerator and denominator larger, while the case \( R_t < 0 \) (anticyclonic flow) results in an increase in the numerator, but a decrease in the denominator. These relations help to explain the fact that the gradient wind is a better approximation than the geostrophic wind in the cycloonic case, but generally not in the anticyclonic case.

8.6 Properties of the Geostrophic Wind

As indicated by the previous sections we may consider the geostrophic wind as a first approximation to the actual wind in the atmosphere above the lowest layer of the troposphere. There is a wide use of the geostrophic wind in many practical problems. Because of the wide use it is of importance to investigate the properties of this wind field.

We shall begin by a consideration of the calculation of the geostrophic wind from synoptic maps. Using height as the vertical co-ordinate we have

\[ \vec{V}_g = \frac{a}{f} \hat{k} \times \vec{v}_p \]  

(8.6.1)

and

\[ V_g = \frac{a}{f} \frac{\delta p}{\delta n_E} \]  

(8.6.2)

The direction of the geostrophic wind is along the isobars, and the magnitude of \( \vec{V}_g \) in a point can be calculated from (8.6.2) where \( \delta p \) is the pressure interval for the isobars, \( \delta n_E \) the normal distance between the isobars on the earth, \( a \) the
specific volume, which can be computed from temperature and pressure, and \( f = 2\Omega \sin \varphi \)
the value of the Coriolis parameter.

In practice we determine \( V_g \) from a map, and it is the distance \( \delta n_M \) which we know. We must therefore introduce \( \delta n_M \) in (8.6.2), and this is done using the map factor \( m \), defined by

\[
m = \frac{\delta n_M}{\delta n_E} \quad \text{(8.6.3)}
\]

We have therefore

\[
V_g = \frac{\Omega m}{f} \frac{\delta x}{\delta n_M} \quad \text{(8.6.4)}
\]

The map factor \( m \) is a function of latitude only for the maps used in meteorological practice, and \( V_g \) may be considered as a function of latitude, distance between the isobars (\( \delta p = \text{const.} \)) and specific volume. In a given surface of constant height (say, mean sea-level) we may with good approximation consider \( a = a_s \), where \( a_s \) is a standard value of \( a \) for that level. \( V_g \) is then only a function of latitude and \( \delta n_M \), and the value of \( V_g \) is in practice determined by a graphical method using a geostrophic wind ruler.

If we were to map the atmosphere using constant height maps we would have to construct a geostrophic wind ruler for each level because \( a \), the specific volume, varies strongly in the vertical direction. However, in meteorological practice we use the height fields of the isobaric surfaces to depict the state of the atmosphere in the upper layers, although the surface map applies to mean sea-level.

For an isobaric surface we have

\[
V_g = \frac{\Omega}{f} \frac{\delta z}{\delta v}
\]

and

\[
V_g = \frac{\Omega}{f} \frac{\delta z}{\delta n_B} \quad \text{(8.6.6)}
\]

Using (8.6.3) we find

\[
V_g = \frac{\Omega m}{f} \frac{\delta z}{\delta n_M} \quad \text{(8.6.7)}
\]
It is seen from (8.6.7) that \( \mathbf{V}_g \) is a function of latitude and the distance between the isohyposes \( \delta n_M \), since \( g \) and \( \delta z \) are constants. We notice therefore that a geostrophic ruler for an isobaric map can be constructed, and that the same ruler can be used for all isobaric surfaces because none of the quantities in (8.6.7) has a height dependence. We may consider this property of (8.6.7), as compared with (8.6.4) where \( \mathbf{a} \) has a strong dependence on height, as one of the major reasons for employing maps of the geopotential in the isobaric surfaces instead of the pressure in level surfaces.

We shall next consider the divergence of the geostrophic wind. From (8.6.5) we find:

\[
\nabla \cdot \mathbf{V}_g = - \frac{g}{f} \frac{\partial G}{\partial x} + \frac{\partial}{\partial y} \left( \frac{g \frac{\partial z}{\partial y}}{f} \right) = - \frac{g}{f^2} \frac{\partial f}{\partial y} \frac{\partial z}{\partial x} \tag{8.6.8}
\]

Using

\[
\nabla^2 \mathbf{V}_g = \frac{g}{f} \frac{\partial z}{\partial x} \tag{8.6.9}
\]

and

\[
\beta = \frac{\partial f}{\partial y} = \frac{\partial (2\Omega \sin\phi)}{a \partial \phi} = \frac{2\Omega}{a} \cos\phi \tag{8.6.10}
\]

we may write

\[
\nabla \cdot \mathbf{V}_g = - \frac{\beta}{f} \mathbf{V}_g = - \frac{V_g}{a \tan\phi} \tag{8.6.11}
\]

where \( \phi \) is latitude.

It is seen from (8.6.11) that geostrophic divergence is found where \( V_g \) is negative, i.e., in the region between the ridge and trough in the geopotential on an isobaric surface, while geostrophic convergence is found where \( V_g \) is positive, i.e., between the trough and the ridge.

Because of the factor \( a \tan\phi \) in (8.6.11) we find that \( \nabla \cdot \mathbf{V}_g \) is very small in middle and high latitudes. It turns out that the divergence of the ageostrophic wind is, in general, much larger than the divergence of the geostrophic wind in spite of the fact that the magnitude of the ageostrophic wind \( \mathbf{V}_a = \mathbf{V} - \mathbf{V}_g \) is much smaller than \( \mathbf{V} \). The quantity \( \nabla \cdot \mathbf{V}_g \) can not therefore be used as an approximation to \( \nabla \cdot \mathbf{V} \). This result is very important in the application of the geostrophic wind especially in the field of numerical weather prediction, and it shows an important
example of the fact that even though the magnitude and direction of a given vector field may be a reasonable approximation to the actual field as observed in the atmosphere, it is not necessarily true that derivatives of that field are also reasonable approximations to the corresponding derivatives of the actual fields.

Let us finally consider the vorticity of the geostrophic wind. We find

\[ \zeta_g = \mathbf{\hat{k}} \cdot (\nabla \times \mathbf{v}_g) = \frac{\partial}{\partial x} \left( \frac{f}{f} \frac{\partial z}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{f}{f} \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} + \frac{\beta}{f} \frac{u_g}{v} \tag{8.6.12} \]

or

\[ \zeta_g = \frac{f}{f} \frac{u_g}{v} + \frac{u_g}{a \tan \phi} \tag{8.6.13} \]

The last term in (8.6.13) is similar to the expression in (8.6.11) except that we now have \( u_g \) replacing \( v_g \). Since \( u_g \) and \( v_g \) are in general of the same order of magnitude, we find that the last term in (8.6.13) will be small in middle and high latitudes. However, the first term in (8.6.13) is not small, in general, and it is for this reason that the vorticity of the geostrophic wind is a good approximation to the vorticity of the actual wind field, or

\[ \zeta \approx \zeta_g \approx \frac{f}{f} \frac{u_g}{v} \tag{8.6.14} \]

(8.6.14) shows clearly that \( \zeta_g \) is positive in a minimum of the geopotential on an isobaric map, and that \( \zeta_g \) is negative in a maximum. Positive values of \( \zeta_g \) (or of \( \zeta \)) are therefore normally called cyclonic and negative values anticyclonic vorticity.

While the neglect of the second term in (8.6.13) is justified in general, there are exceptions. In the region of the subtropical jetstream we very often find large values of \( u_g \) in the upper troposphere. It is furthermore known that strong horizontal shear zones are found on the south side of the subtropical jetstream. In this region we will therefore find anticyclonic vorticity, and individual cases have been found where the anticyclonic vorticity is almost as large as the Coriolis parameter which is equal to \( 6.2 \times 10^{-5} \text{s}^{-1} \) at \( 25^\circ \text{N} \). The value of the last term in (8.6.13) is equal to \( 3.4 \times 10^{-5} \text{s}^{-1} \) if \( u_g = 100 \text{ m s}^{-1} \) at that latitude. It is therefore seen that the term is not negligible in such a case.
8.7 The Geostrophic Thermal Wind

We shall in this section investigate the variation of the geostrophic wind with respect to height. It is most convenient to use the representation of the geostrophic wind given in (8.6.5) referring to the geopotential of the isobaric surfaces. Differentiating (8.6.5) with respect to \( p \) we get

\[
\frac{\partial \vec{v}}{\partial p} = \frac{1}{f} \vec{k} \times \nabla \left( \frac{\partial \Phi}{\partial p} \right)
\]  

(8.7.1)

or

\[
\frac{\partial \vec{v}}{\partial z} = \frac{\partial p}{\partial z} \frac{\partial \vec{v}}{\partial p} = + \frac{f}{a} \frac{1}{f} \vec{k} \times \nabla a
\]  

(8.7.2)

where we have used the hydrostatic relation

\[
\frac{\partial \Phi}{\partial p} = -a
\]  

(8.7.3)

Since the gradient appearing in (8.7.2) is evaluated with constant pressure \( p \) we have

\[
\frac{1}{a} \nabla a = \frac{1}{T} \nabla T = \frac{1}{\Theta} \nabla \Theta
\]  

(8.7.4)

and we may write:

\[
\frac{\partial \vec{v}}{\partial z} = \frac{f}{a} \vec{k} \times \nabla a = \frac{f}{a} \vec{k} \times \frac{\nabla T}{T} = \frac{f}{a} \vec{k} \times \frac{\nabla \Theta}{\Theta}
\]  

(8.7.5)

The most useful of the formulae in (8.7.5) is probably the middle one which shows that the wind increase per unit distance is determined by the horizontal gradient of the temperature in the isobaric surface, and in such a way that the vector \( \frac{\partial \vec{v}}{\partial z} \) is directed along the isotherm with the cold air to the left (see Figure 8.8).
It is for this reason that any of the expressions (8.7.5) is called the thermal wind equation, and $\frac{\partial \mathbf{v}}{\partial z}$ is called the thermal wind. We see that the thermal wind has the same type of relation to the isotherms in an isobaric surface as the geostrophic wind has to the isohypses in the same surface.

The thermal wind equation is very important because we may calculate the wind increase in the vertical direction from the temperature gradient or vice versa. As an example we may mention the hodograph which is a curve obtained in the following way. The wind observations obtained during an ascent are plotted with a common origin (see Figure 8.9). The curve through the endpoints of the wind vectors is the hodograph. We see that the thermal wind $\frac{\partial \mathbf{v}}{\partial z}$ is the tangent to the hodograph.

![Figure 8.9](image-url)

It follows then that the isotherms have the same direction as $\frac{\partial \mathbf{v}}{\partial z}$ if we can assume that $\frac{\partial \mathbf{v}}{\partial z} \approx \frac{\partial \mathbf{v}}{\partial z}$. The lines marked $T-\delta T$ and $T$ are isotherms drawn according to the geostrophic thermal wind. From the direction of the isotherms we may naturally find the direction of $\nabla_T$, and we may compute that quantity from

$$\left| \nabla_T \right| = \frac{g}{\mathbf{v}} \left| \frac{\partial \mathbf{v}}{\partial z} \right|$$

(8.7.6)

The thermal wind and the hodograph are also very important in computing the temperature advection assuming that the wind is geostrophic. The geostrophic temperature advection is defined as $-\frac{\mathbf{v}}{g} \cdot \nabla_T$. We may think of the advection as the temperature change which would take place at a given point if the flow were adiabatic and horizontal. The thermodynamic equation in this case is
\[ \left( \frac{dT}{dt} \right)_{\text{adv}} = \nabla \cdot \nabla_T T \]  \hspace{1cm} (8.7.7)

From (8.7.5) and Figure 8.8 we obtain
\[ \nabla_T T = - \frac{f}{g} \mathbf{T} \times \frac{d\mathbf{v}}{dz} \]  \hspace{1cm} (8.7.8)

Substituting from (8.7.8) in (8.7.7) we get
\[ \left( \frac{dT}{dt} \right)_{\text{adv}} = \frac{f}{g} \mathbf{T} \cdot \left( \mathbf{k} \times \frac{d\mathbf{v}}{dz} \right) \]  \hspace{1cm} (8.7.9)

which according to the rules of vector algebra is
\[ \left( \frac{dT}{dt} \right)_{\text{adv}} = - \frac{f}{g} \mathbf{T} \cdot \left( \mathbf{v}_g \times \frac{d\mathbf{v}}{dz} \right) \]  \hspace{1cm} (8.7.10)

Equation (8.7.10) is well suited to calculate the temperature advection from the hodograph.

![Hodograph Diagram]

Figure 8.10

Figure 8.10 shows two typical cases. In the left-hand figure we have a case where the horizontal wind turns to the right with height. We find from (8.7.10) that \( \left( \frac{dT}{dt} \right)_{\text{adv}} \) is positive. This case is called warm-air-advection because warmer
air is replacing colder air at point A due to the advection term. The right-hand
figure in Figure 8.10 shows the case where the horizontal, geostrophic wind turns to
the left with height. We find from (8.7.10) that $(\partial T/\partial t)_{adv} < 0$ because the vector
$\vec{v}_g \times (\partial \vec{v}_g/\partial z)$ is directed outwards from the plane of the paper. It is thus directed
in the same direction as $\vec{k}$. It is also obvious from the figure that this case
 corresponds to cold-air-advection.

In order to use (8.7.10) in practice it is quite often necessary to replace
$\partial \vec{v}_g/\partial z$ by a finite difference over a distance $h$. We have then
\[
\frac{\partial \vec{v}_g}{\partial z} \approx \frac{1}{h}(\vec{v}_g(z + h) - \vec{v}_g(z)) \quad (8.7.11)
\]
and we get
\[
(\frac{\partial T}{\partial t})_{adv} = - \frac{f}{g} \frac{T}{h} \vec{k} \cdot \vec{v}_g(z) \times \vec{v}_g(z + h) \quad (8.7.12)
\]

The term $\vec{k} \cdot \vec{v}_g(z) \times \vec{v}_g(z + h)$ is the numerical value of the area of a
parallelogram with the sides $\vec{v}_g(z)$ and $\vec{v}_g(z + h)$, and it is seen that $(\partial T/\partial t)_{adv}$ is
proportional to the area swept by the wind vector $\vec{v}_g$ on the hodograph. That area
shall be counted positive if $\vec{v}_g$ turns to the left with height and negative if to
the right. We can explore this idea a little further by introducing the area
\[
A(z) = \frac{1}{2} \vec{k} \cdot \int_0^z \vec{v}_g \times d\vec{v}_g \quad (8.7.13)
\]
where $A(z)$ is the area of the region bounded by $\vec{v}_g(0), \vec{v}_g(z)$ and the hodograph (see
Figure 8.11) and where $A(z)$ will be positive when $\vec{v}_g(z)$ turns to the left with
height. We then have
\[
\frac{dA}{dz} = \frac{1}{2} \vec{k} \cdot \left( \vec{v}_g \times \frac{\partial \vec{v}_g}{\partial z} \right) \quad (6.7.14)
\]
and
\[
(\frac{\partial T}{\partial t})_{adv} = - \frac{2}{g} \frac{T}{T} \frac{dA}{dz} \quad (8.7.15)
\]

Since we have
\[
\frac{1}{a} \left( \frac{\partial a}{\partial t} \right)_{p, adv.} = \frac{1}{T} \left( \frac{\partial T}{\partial t} \right)_{p, adv.} \quad (8.7.16)
\]
it follows from (8.7.15) that
\[
\left( \frac{da}{dt} \right)_{\text{adv.}} = -2 \frac{f}{g} \frac{a}{dz} = 2f \frac{dA}{dp} \tag{8.7.17}
\]

and we obtain, using the hydrostatic equation:

\[
\frac{d}{dp} \left( \frac{d\Phi}{dt} \right)_{\text{adv.}} = -2f \frac{dA}{dp} \tag{8.7.18}
\]

which upon integration gives:

\[
\varepsilon \left( \frac{d}{dt} (z - z_1) \right) = -2f (A_2 - A_1) \tag{8.7.19}
\]

which indicates that the change in thickness is proportional to the change in area from level 1 to level 2.

![Figure 8.11](image)

8.8 The Thermal Wind and the Solenoidal Vector

We will show in this section that there is a connexion between the thermal wind and the solenoidal vector introduced in section 6.5. We had

\[
\vec{S} = -V_a \times \nu_p \tag{8.8.1}
\]

The horizontal component of \( \vec{S} \) is

\[
\vec{S}_h = - \left\{ \begin{pmatrix} \frac{d}{dy} \nu_p - \frac{d}{dx} \nu_p \\ \frac{d}{dz} \nu_p \nu_p \end{pmatrix} \nu_p^* - \left( \begin{pmatrix} \frac{d}{dx} \nu_p - \frac{d}{dz} \nu_p \nu_p \end{pmatrix} \nu_p^* \right) \right\} \tag{8.8.2}
\]

Using the hydrostatic equation and the conversion formulae (5.4.7) and (5.4.8) to the \( p \)-system we get

\[
\vec{S}_h = \varepsilon_p \left\{ \begin{pmatrix} \frac{d}{dy} \\ \frac{d}{dx} \nu_p \end{pmatrix} \nu_p^* - \left( \begin{pmatrix} \frac{d}{dx} \nu_p \nu_p \end{pmatrix} \nu_p^* \right) \right\} = -\varepsilon_p \nu_p \times \nu_p = -\varepsilon_p \nu_p \times \frac{V_p a}{a} \tag{8.8.3}
\]

When we introduce (8.8.3) in the thermal wind equation we get
\[
\frac{d \vec{v}}{dz} = -\frac{1}{f} \vec{g}_h
\] (8.8.4)

which shows that the thermal wind is proportional to the horizontal component of the solenoidal vector.

8.9 Changes in Static Stability due to Advection

There are many measures of static stability of which the most common is the lapse rate. We shall in this section use a measure which is

\[
s = g \frac{d \ln \theta}{dz}
\] (8.9.1)

Let us first show that \(s\) is proportional to the difference between the dry adiabatic and the actual lapse rate. We find from the definition of the potential temperature that

\[
\vec{s} = \frac{g}{T} (\gamma_d - \gamma)
\] (8.9.2)

where \(\gamma_d = g/c_p\) and \(\gamma = -dT/dz\).

It follows furthermore from the definition of the potential temperature that

\[
\frac{1}{T} \nabla_p T = \frac{1}{\theta} \nabla_p \theta
\] (8.9.3)

and

\[
\frac{1}{T} \left( \frac{dT}{dt} \right)_p = \frac{1}{\theta} \left( \frac{d \theta}{dt} \right)_p
\] (8.9.4)

Introducing these results in (8.7.15) we find that

\[
\left( \frac{1}{\theta} \frac{d \theta}{dt} \right)_{\text{adv.}} = -2 f \frac{\nabla A}{g \frac{d \theta}{dz}}
\] (8.9.5)

from which it follows that

\[
\left( \frac{d S}{dt} \right)_{\text{adv.}} = -2 f \frac{\nabla A}{g \frac{d^2 A}{dz^2}}
\] (8.9.6)

It is thus seen from (8.9.6) that the hodograph can be used to find both the changes in temperature and the changes in stability due to advection. Let us consider some examples. The first is shown schematically in Figure 8.12.
Since \( \vec{v}_g \) turns to the right we have that \( A \) is negative and thus \( da/dz > 0 \). It is also seen that the area \((A_{34})\) is numerically larger than the area \((A_{23})\), but since both are negative, we find that \( d^2a/dz^2 < 0 \), and thus \((ds/\partial t)_{adv} > 0 \). The situation depicted in Figure 8.12 can thus be characterized by the words: warm-air-advection, increasing with height, stabilizing.

Consider next the hodograph in Figure 8.13. We have, first of all, cold-air-advection, which is increasing with height \((A_{01}) < (A_{12}) < (A_{23})\), but it is also seen that \( d^2a/dz^2 > 0 \). We have therefore a decrease in \( s \), and the situation is destabilizing.
The reader is invited to analyse the two hodographs shown in Figure 8.14.

![Image of Figure 8.14]

Figure 8.14

The last example is shown schematically in Figure 8.15. The solid lines are the isobars on the surface map showing a wave cyclone with a **cold front** and a **warm front**. The dashed lines are the isohypses of some upper-air map, say 500 mb.
Point C has recently been passed by the cold front. The hodograph for point C is shown schematically in Figure 8.16.

Since $\vec{v}$ is turning to the left with height we have $A > 0$. It is also seen that $dA/dz > 0$, which means cold-air-advection. Furthermore, $d^2A/dz^2 > 0$, which according to (8.9.6) means destabilization.

Consider next point B, which is located ahead of the warm front. The hodograph in point B is shown schematically in Figure 8.17.

At point B we have $A < 0$ and $dA/dz < 0$ and therefore warm-air-advection. Since furthermore $d^2A/dz^2 < 0$ we find a tendency for stabilization at point B.
Chapter IX

THE STATIONARY CIRCULAR VORTEX

9.1 **Introduction**

The purpose of this chapter is to investigate in some detail the stationary circular vortex. We have already considered a special case of such a vortex in section 8.4 when we investigated the gradient wind relation, but in that case, we considered the relative motion of a particle moving in a circular trajectory on the earth, and we restricted ourselves to horizontal motion only.

The situation which we shall investigate in this chapter is more general. We consider an atmosphere where all particles rotate (in circular orbits) around a common axis of rotation in an absolute co-ordinate system. It will be assumed that the motion is axisymmetric, i.e. that the motion is the same in all planes containing the axis of rotation. It is then sufficient to consider only one of these planes (see Figure 9.1).

![Figure 9.1](image-url)
The rotation vector $\vec{\Omega}$ points along the axis of rotation in Figure 9.1. The circle in a plane perpendicular to $\vec{\Omega}$ is the orbit of an arbitrary particle. The position of a particle is determined by its height $z$ above the lower plane, the distance $R$ from the axis of rotation, and, in general, the angle $\lambda$ (cylindrical co-ordinate). In our case we need only $R$ and $z$ because of the assumption of symmetry.

The field of motion may be described by the angular velocity $\Omega = \Omega(R,z)$, the linear velocity $V = OR = V(R,z)$ or the "spin" $S = RV = OR^2$. The spin is naturally the moment of the linear velocity around the axis of rotation.

The forces acting on an arbitrary particle are the pressure force, the gravitational force and the centrifugal force. Since the motion is considered stationary in the absolute framework we have

$$-\sigma Vp - \psi_a - \vec{\Omega} \times (\vec{\Omega} \times \vec{R}) = 0 \quad (9.1.1)$$

as the equation of motion.

The centrifugal force is directed outwards and may be written

$$-\vec{\Omega} \times (\vec{\Omega} \times \vec{R}) - \Omega^2 VR \quad (9.1.2)$$

where $VR$ is a unit vector in the direction of the radius vector, i.e. $\vec{T} = VR$. The unit vector in the direction normal to the meridional plane is $\vec{J} = RV\lambda$, and in the vertical direction $\vec{K} = Vz$. We may therefore write $(9.1.1)$ in the form:

$$-\sigma Vp - \psi_a + \Omega^2 VR = 0 \quad (9.1.3)$$

We shall assume in the following that the geopotential $\psi_a = \psi_a(R,z)$ is known. The forces in balance are shown in Figure 9.2.

9.2 Balance Requirements

Equation $(9.1.3)$ expresses the balance between the forces in the motion under consideration. We shall now obtain the vorticity equation corresponding to $(9.1.3)$. We note then that the centrifugal force can be written in the form

$$\Omega^2 VR = \Omega^2 V(\frac{1}{2} R^2) \quad (9.2.1)$$
We then apply the operator $\nabla x$ to (9.1.3) and we obtain

$$\nabla x (-\nabla p) + \nabla \Omega^2 \times \nabla \left(\frac{1}{2} R^2\right) = 0$$  \hspace{1cm} (9.2.2)

which may also be written as

$$\nabla \Omega^2 \times \nabla \left(\frac{1}{2} R^2\right) = \nabla x (-\nabla p) = \vec{S}$$  \hspace{1cm} (9.2.3)

where the right-hand side is the solenoidal vector. Equation (9.2.3) is important because it shows that the four vectors $\nabla \Omega^2$, $\nabla \left(\frac{1}{2} R^2\right)$, $\nabla x$ and $-\nabla p$ have to satisfy (9.2.3) and are therefore dependent on each other. The vector $\nabla \left(-\frac{1}{2} R^2\right) = -R \nabla R$ will always point in towards the axis of rotation. We may otherwise distinguish between three separate situations:

(a) **Beroclinic vortex with cold centre**

We consider Figure 9.3. With the relative position of $\nabla x$ and $-\nabla p$ we can deduce that if we move along the isobaric surface $p = \text{const.}$ towards the axis of rotation we will experience smaller values of $\omega$ which then for constant values of $p$ correspond to smaller values of $T$, which in turn means a cold centre in the isobaric surface. The solenoidal vector will point outwards from the figure, and it follows then that $\nabla \Omega^2$ must have a position as indicated in Figure 9.3 in order that the cross-product on the left-hand side of (9.2.3) also will point towards the reader.
In this case we have by definition that $\vec{S} = \nabla \alpha \times (-\nabla p) = 0$ which means that $\nabla \alpha$ and $-\nabla p$ are directed in the same direction. It follows then that $\nabla \Omega^2$ will point in the same direction as $\nabla(-\frac{1}{2}R^2)$ as shown in Figure 9.4.
(c) \textit{Preclonic vortex with warm centre}

As indicated in Figure 9.5 we will in this case experience larger values of $a$ as we move inwards towards the axis of rotation along the isobaric surface. The centre is therefore warm (large values of $T$). The solenoidal vector will be directed into the paper, and it is necessary that $\nabla \Omega^2$ has a direction as seen on Figure 9.5 in order that $\nabla \Omega^2 \times \nabla \left(-\frac{1}{2} \Omega^2 \right)$ also points in the same direction.

In the interpretation of Figures 9.3, 9.4 and 9.5, it is very important to remember that we are dealing with absolute motion. $\Omega$ is thus the angular velocity of the absolute motion, but we can obtain information about the relative motion in the following way. Since

$$\Omega = \Omega_E + \Omega_R \quad (9.2.4)$$

where $\Omega_E$ is the angular velocity of the earth and therefore constant, we find that $\nabla \Omega = \nabla \Omega_R$ and therefore $\nabla \Omega^2 = 2 \Omega \nabla \Omega_R$. It is however easier to discuss the implications of (9.2.3) if we express it in scalar form. Using the rules of the vector product we find that

$$- R \frac{\partial \Omega^2}{\partial z} = \frac{\partial a}{\partial R} \frac{\partial p}{\partial z} - \frac{\partial a}{\partial z} \frac{\partial p}{\partial R} \quad (9.2.5)$$

The right-hand side of (9.2.5) may be written in a simpler form by noting that the basic transformation formula from the $z$-system to the $p$-system is

$$\left( \frac{\partial b}{\partial R} \right)_z = \left( \frac{\partial b}{\partial R} \right)_p - \frac{\partial b}{\partial z} \left( \frac{\partial z}{\partial R} \right)_p \quad (9.2.6)$$

Using (9.2.6) we get

$$\frac{\partial a}{\partial R} \frac{\partial p}{\partial z} - \frac{\partial a}{\partial z} \frac{\partial p}{\partial R} = \left( \frac{\partial a}{\partial R} \right)_p - \frac{\partial a}{\partial z} \left( \frac{\partial z}{\partial R} \right)_p \left( \frac{\partial p}{\partial z} \left( \frac{\partial a}{\partial R} \right)_p \right) \quad (9.2.7)$$

(9.2.5) may therefore be written

$$\frac{\partial \Omega^2}{\partial z} = - \frac{1}{R} \frac{\partial p}{\partial z} \left( \frac{\partial a}{\partial R} \right)_p \quad (9.2.8)$$

We can again consider the three cases in the light of (9.2.8), where $(\partial a/\partial R)_p$ is the variation of $a$ in the isobaric surface in the radial direction.
(i) **Baroclinic vortex with a cold centre**

As can be seen from Figure 9.3 and the previous discussion we have \((\partial \alpha / \partial R)_p > 0\) in this case. Assuming that \((\partial p / \partial z) < 0\), as we shall throughout this discussion, we find that \((\partial \Omega^2 / \partial z) > 0\) indicating an increase in the angular velocity along the lines parallel to the axis of rotation.

(ii) **Barotropic vortex**

In this case we have by definition that \((\partial \alpha / \partial R)_p = 0\) since \(\alpha\) is a function of \(p\) only. It follows from (9.2.8) that \((\partial \Omega^2 / \partial z) = 0\). The equiscalar surfaces for \(\Omega\) are thus cylindrical surfaces with the axis coinciding with the axis of rotation. We may also express the result in the form that \(\Omega\) is a function of \(R\) only in the barotropic case.

(iii) **Baroclinic vortex with a warm centre**

As is seen from Figure 9.5 we have \((\partial \alpha / \partial R) < 0\) and therefore \((\partial \Omega^2 / \partial z) < 0\), and it follows that the angular velocity will decrease along the lines parallel to the axis of rotation.
The previous results are quite general and are applicable to any circular vortex in which we have a balance between the gravitational force, expressed as the gradient of a geopotential, the pressure force and the centrifugal force, or, in other words, a system for which (9.1.1) applies in the absolute framework.

We can apply the results to various systems. In the case of the earth's atmosphere the natural application is to the circumpolar vortex which we obtain by averaging all fields along the latitude circles. This is done mathematically by using the operator
\[
(\ )_z = \frac{1}{2\pi} \int_0^{2\pi} (\ ) \, d\lambda
\]  
(9.2.9)

where \(\lambda\) is longitude. Each scalar quantity is, after averaging, a function of latitude and pressure (or height above mean sea-level).

We can easily convert from the co-ordinates of height \((h)\) above mean sea-level and latitude \((\varphi)\) to \(R\) and \(Z\) as applied in this chapter. Using Figure 9.6 where \(\phi\) is measured from the equatorial plane we get
\[
R = (a + h) \cos \varphi
\]
\[
Z = (a + h) \sin \varphi
\]  
(9.2.10)

where \(a\) is the radius of the earth.

![Figure 9.6](image-url)
As known from observations we have a cyclonic circumpolar vortex in most parts of the troposphere in the relative motion. Since the relative motion is slow compared to the rotation of the earth we have a cyclonic vortex everywhere in the absolute motion. Observations show further that the circumpolar vortex is cold at the North Pole and warm at the Equator. From the result obtained here we may conclude that the absolute angular velocity will increase along the lines parallel to \( \Omega \) in Figure 9.6, assuming that the zonally averaged circumpolar vortex can be considered as stationary, i.e. no acceleration apart from the centripetal acceleration.

9.3 The Shape of the Isobaric Surfaces

Another application of the analysis made here is the calculation of the shape of the isobaric surfaces in the circumpolar vortex. In order to obtain the solution to this problem we return to equation (9.1.3) and multiply this equation by an arbitrary vector \( \mathbf{dr} \) in the meridional plane. Since, in general,

\[
\mathbf{db} - \nabla b \cdot \mathbf{dr} \tag{9.3.1}
\]

(see section 1.7) we get

\[
-\alpha d\mathbf{p} - d_a + \Omega^2 R dR = 0 \tag{9.3.2}
\]

This equation can be used to obtain information on the shape of the isobaric surfaces in the circumpolar vortex because when we integrate (9.3.2) from \( R = R_0 \) along the isobaric surfaces, i.e. \( d\mathbf{p} = 0 \), we get

\[
\Phi_a (R_0) = \Phi_a (0) + \int_0^{R_0} \Omega^2 R dR \tag{9.3.3}
\]

where \( \Phi_a (0) \) is the geopotential at \( R = 0 \). We have thus

\[
\Delta \Phi_a = \Phi_a (R_0) - \Phi_a (0) = \int_0^{R_0} \Omega^2 R dR \tag{9.3.4}
\]

Either of the equations (9.3.3) or (9.3.4) may be considered as the equation for the isobaric surface in the circumpolar vortex. We may compute the shape if the wind distribution, i.e. \( \Omega = \Omega(R, \varphi) \), is given. Since we can assume with good approximation that

\[
\Phi_a = g_a h + \text{const.} \tag{9.3.5}
\]

we can look upon \( \Delta \Phi_a = g_a \Delta h \) as proportional to the increase in height of the isobaric surface above the geopotential surface \( \Phi_a (0) \) (see Figure 9.7), or
\[ g_a \Delta h = \int_0^R \Omega^2 R \, dR \] \hspace{1cm} (9.3.6)

In Figure 9.7, \( h_0 \) indicates the height of the isobaric surface above mean sea-level. We have \( \Phi_a(0) = g_a h_0 + \text{const.} \). The main result, obtained from (9.3.6) and indicated on Figure 9.7 is that \( \Delta h \) is positive for all values of \( R \), or, in other words, the isobaric surfaces in the circumpolar vortex slopes in towards the axis of rotation relative to the equipotential surfaces. We can of course equally well say that the pressure decreases towards the pole on the equipotential surfaces.

In using (9.3.6) we shall first consider the case where \( \Omega = \Omega_0 = \text{const.} \), which may be considered as the form of the isobaric surfaces when there is no relative motion. We get

\[ g_a \Delta h_E = \frac{1}{2} \Omega^2 R_0^2 \] \hspace{1cm} (9.3.7)

It may be of interest to estimate \( \Delta h_E \) where it is largest, i.e., at the equator when \( R_0 \approx a = \text{the radius of the earth} \). Using \( \Omega_0 = 7.3 \times 10^{-5} \text{s}^{-1} \), \( g_a \approx 10 \text{ m s}^{-2} \) and \( R_0 \approx (2 \times 10^7 \text{m}) \) m we find \( (\Delta h_E)_{\text{max}} \approx 10^4 \text{m} = 10 \text{ km} \).

The surface \( h_0 + \Delta h_E \) will thus have a slope as indicated by the dashed curve on Figure 9.7 although the figure is out of proportion since \( (\Delta h_E)_{\text{max}} \) is much too large compared to \( a \), the radius of the earth.

It is of interest to note that the equation for the dashed curve is

\[ z^2 = (a + h_0 + \Delta h_E)^2 - R_0^2 \] \hspace{1cm} (9.3.8)
or

\[ z^2 = \left( a^o + \frac{1}{2} G^o E \frac{R^2}{E^o} \right)^2 - R^o \]

(9.3.9)

where (9.3.8) is obtained from the triangle (OAB) in which OB = R^o, OA = a^o + h^o + A h^E

and AB = z. In (9.3.9) we have introduced a^o = a + h^o. It is thus possible to
calculate the exact slope of the curve z = z(a^o, G^o E, G^o E^o, R^o) as determined by (9.3.9).

We now note that the curve determined by (9.3.9) is approximately parallel
to the shape of the earth. When we therefore consider the relative motion, we do so
relative to the surfaces determined by (9.3.9) or, equivalently, (9.3.7). If we now
introduce \( \Omega = \Omega^E + \Omega^R \) in the general expression (9.3.6) we find

\[ g^a \Delta h = g^a \Delta h^E + 2 \Omega^E \int_0^R \Omega^R \frac{R dR}{R} + \int_0^R \Omega^2 \frac{R dR}{R} \]

(9.3.10)

or

\[ g^a \Delta (h - h^E) = 2 \Omega^E \int_0^R \Omega^R \frac{R dR}{R} + \int_0^R \Omega^2 \frac{R dR}{R} \]

(9.3.11)

We may use (9.3.11) to compute the position of the isobaric surfaces
relative to the surfaces when the relative motion is zero. In order to use (9.3.11)
we must specify \( \Omega^R \) as a function of \( R \) and \( z \). It is seen that the last term in
(9.3.11) always gives a positive contribution, while the first term (the Coriolis
effect) may be positive or negative depending on the sign of \( \Omega^R \).

As a relatively simple example we may consider \( \Omega^R \) given by

\[ \Omega^R = 4 \Omega^E \frac{R}{a} \left( 1 - \frac{R}{a} \right) \]

(9.3.12)

where \( a \) is the radius of the earth. \( \Omega^R \) is zero at \( R = 0 \) and \( R = a \) and has a maximum
value of \( \Omega^E \) at \( R = (1/2)a \). We may consider \( \Omega^R \) as a function of the latitude \( \varphi \) on the
earth since \( R = a \cos \varphi \). The maximum \( \Omega^R \) occurs therefore at \( \varphi = 60^0 \). Substitution of
(9.3.12) in (9.3.11) gives

\[ g^a \Delta (h - h^E) = 8 \Omega^E \frac{R}{a} a^2 \left( \frac{1}{3} \left( \frac{R}{a} \right)^3 - \frac{1}{4} \left( \frac{R}{a} \right)^4 \right) \]

\[ + 16 \Omega^2 \frac{R}{a} a^2 \left( \frac{1}{4} \left( \frac{R}{a} \right)^4 - \frac{2}{5} \left( \frac{R}{a} \right)^5 + \frac{1}{6} \left( \frac{R}{a} \right)^6 \right) \]

(9.3.13)
The value of the first term at the equator, \( R_o = a \), is \( \frac{2}{3} \Omega \Omega_o a^2 \), while the second is \( \frac{4}{15} \Omega \Omega_o a^2 \). Of these two the first is by far the more important since \( \Omega \gg \Omega_o \) in the atmosphere. For \( \Omega_o = 2 \times 10^{-6} \text{s}^{-1} \) which corresponds to a maximum linear velocity of 50 m s\(^{-1}\), we find that the Coriolis contribution is 100 times larger than the contribution of the centrifugal force for the relative motion.

9.4 Rotational Stability

Let us consider the system analyzed in the preceding sections once again, but with the simplification that the atmosphere is barotropic such that \( a = a(p) \). According to (9.2.9) \( \Omega \) is a function of \( R \) only in such a system. Since we are considering an absolute system we have conservation of the circulation along a physical curve.

We now consider a curve which is a circle with its centre on the axis of rotation. The circulation is

\[
C = 2\pi R \Omega R = 2\pi \Omega R^2
\]  

(9.4.1)

The problem which we want to consider is the following: Imagine that we let all the particles on the circle move into a new circle through all receiving the same displacement in the meridional planes in which they are situated. Will the particles tend to return to their original positions or will they tend to be removed further from their original locations? In the first case we shall call the vortex stable, in the second case unstable, while the vortex will be called neutral if the displaced particles are in equilibrium in their new surroundings.

The key to the solution is the conservation of \( C \), defined in (9.4.1). Suppose that the particles are initially at \( R = R_1 \). We have then a circulation \( C_1 = 2\pi \Omega R_1^2 \). Assume next that the particles all move to occupy a new circle with radius \( R_2 > R_1 \). The pressure and the gravitational forces at the new position, i.e., \(-a V p\)_2 and \(-a V \phi\)_2, which will now act on the displaced particles have a resultant force which according to Figure 9.2 is

\[
-\frac{\Omega^2}{2} R_2 V R
\]  

(9.4.2)

The third force acting on the displaced particles is the centrifugal force. This force has the form

\[
+\Omega^2 R_2 V R
\]  

(9.4.3)
where $\Omega_*$ is the angular velocity of the displaced particles. $\Omega_*$ for the particles may be computed from the conservation of $C$, i.e.

$$ C_1 = 2\pi \Omega_1^2 R_1^2 = 2\pi \Omega_*^2 R_2^2 \quad (9.4.4) $$

giving

$$ \Omega_* = \frac{C_1}{2\pi R_2^2} \quad (9.4.5) $$

The net force acting on the displaced particles in the outward direction is

$$ \left(\Omega_*^2 - \Omega_2^2\right) R_2 V_R \quad (9.4.6) $$

We will thus have stability, if

$$ \Omega_*^2 < \Omega_2^2 \quad (9.4.7) $$

or

$$ C_1 > C_2 \quad (9.4.8) $$

Suppose on the other hand that we had displaced the particle to a smaller radius $R_2$. According to (9.4.6) we would then have stability, if

$$ \Omega_*^2 > \Omega_2^2 \quad (9.4.9) $$

or

$$ C_1 < C_2 \quad (9.4.10) $$

where subscript 1 in each case is the original position and subscript 2 the displaced position. We can thus combine (9.4.8) and (9.4.10) into a single statement, saying that the autobarotropic circumpolar vortex is stable if $(\partial C^2/\partial R) > 0$, neutral if $(\partial C^2/\partial R) = 0$, and unstable if $(\partial C^2/\partial R) < 0$. This statement is the condition for rotational stability.

According to the result just obtained we see that the vortex is stable if $\Omega = \text{const.}$ because

$$ \frac{\partial C^2}{\partial R} = \frac{\partial \left[(2\pi \Omega R^2)^2\right]}{\partial R} = 4\pi^2 \Omega^2 (4R^2) > 0 \quad (9.4.11) $$

The vortex is neutral if $C = 2\pi \Omega R^2 - 2\pi V_a R = \text{const.}$, since in that case we have $\partial C^2/\partial R = 0$. In the neutral vortex we have a linear wind which is inversely proportional to the distance from the centre. While such a vortex is possible in general, it can not hold for $R \rightarrow 0$ since $V_a$ would be infinitely large.
The criterion for stability may also be expressed in terms of the vorticity. In this case we must first find the vorticity for the general flow under consideration. This can be done by using the theorem that the circulation around an infinitesimal area is equal to the vorticity times the area. In order to define the area we consider an arbitrary vector \( \delta \mathbf{r} = \delta \mathbf{f} \) in one of the planes containing the axis of rotation. When \( \delta \mathbf{r} \) is turned to angle \( \delta \lambda \) around the axis of rotation it will occupy the position \( P'Q' \). The area under consideration is \( PQQ'P' \), and the area vector is

\[
\delta \mathbf{A} = \delta \mathbf{r} \times R \delta \lambda \mathbf{j}
\]  

(9.4.12)

where \( \mathbf{j} \) is a unit vector normal to the meridional plane.

The circulation is

\[
\delta C = \mathbf{v} \cdot R \delta \mathbf{A} = \mathbf{v} \cdot R \delta \lambda \mathbf{j} = \delta (\mathbf{V} \cdot \mathbf{R}) \delta \lambda
\]  

(9.4.13)

which may also be written

\[
\delta C = (\mathbf{v} \cdot \mathbf{R}) \delta \mathbf{r} \delta \lambda
\]  

(9.4.14)

Denoting the vorticity \( \mathbf{\Omega} \times \mathbf{V} \) we have

\[
\mathbf{v} \cdot \mathbf{R} \delta \mathbf{r} \delta \lambda = (\mathbf{\Omega} \times \mathbf{V}) \cdot (\delta \mathbf{r} \times R \delta \lambda \mathbf{j})
\]  

(9.4.15)

or

\[
\frac{1}{R} \mathbf{v} \cdot \mathbf{R} \delta \mathbf{r} = \delta \mathbf{r} \cdot (\mathbf{j} \times (\mathbf{v} \times \mathbf{V}))
\]  

(9.4.16)

Since (9.4.16) must be true for all vectors \( \delta \mathbf{r} \) we have

\[
\mathbf{j} \times (\mathbf{v} \times \mathbf{V}) = \frac{1}{R} \mathbf{v} \cdot \mathbf{R}
\]  

(9.4.17)

or finally,
\[ \mathbf{v} \times \mathbf{v}_a = \frac{1}{\mathbf{R}} \mathbf{v}(\mathbf{R}) \times \mathbf{j} \quad (9.4.16) \]

In order to obtain (9.4.10) from (9.4.17) we must note that since \( \mathbf{v} = \mathbf{v}(\mathbf{R}, z) \), \( \mathbf{v}(\mathbf{R}) \) will be a vector in the meridional plane. It follows that since \( \mathbf{j} \) is normal to the plane, then \( \mathbf{v} \times \mathbf{v} \) must be in the meridional plane. From (9.4.13) we can further deduce that \( \mathbf{v} \times \mathbf{v} \) is directed along the isolines for \( \mathbf{R} = \Omega \mathbf{R}^2 \).

We shall now show that the stability criterion may also be written

\[ \mathbf{\Omega} \cdot (\mathbf{v} \times \mathbf{v}_a) > 0 \quad (9.4.19) \]

The reason is that

\[
\mathbf{\Omega} \cdot (\mathbf{v} \times \mathbf{v}_a) = \frac{1}{\mathbf{R}} \mathbf{\Omega} \cdot (\mathbf{v}(\mathbf{R}) \times \mathbf{j})
= \frac{1}{2\pi \mathbf{R}} (\mathbf{\Omega} \mathbf{k}) \cdot \left[ \left( \frac{\partial C}{\partial \mathbf{R}} \mathbf{\hat{r}} + \frac{\partial C}{\partial z} \mathbf{\hat{z}} \right) \times \mathbf{j} \right]
= \frac{1}{2\pi \mathbf{R}} (\mathbf{\Omega} \mathbf{k}) \cdot \left[ \left( \frac{\partial \mathbf{C}}{\partial \mathbf{R}} \mathbf{\hat{r}} - \frac{\partial \mathbf{C}}{\partial z} \mathbf{\hat{z}} \right) \right]
= \frac{1}{2\pi \mathbf{R}} \mathbf{\Omega} \frac{\partial \mathbf{C}}{\partial \mathbf{R}} = \frac{2\mathbf{C}}{8\pi^2 \mathbf{R}} \frac{\partial \mathbf{C}}{\partial \mathbf{R}} = \frac{1}{2\pi \mathbf{R}} \frac{\partial^2 \mathbf{C}}{\partial \mathbf{R}^2} \quad (9.4.20)
\]

According to (9.4.19) we can state that the condition for stability is that the scalar product of \( \mathbf{\Omega} \) and \( \mathbf{v} \times \mathbf{v}_a \) is positive, but in the autobarotropic case we have that \( \mathbf{v}_a \mathbf{R} = \mathbf{\Omega} \mathbf{R}^2 \) is a function of \( \mathbf{R} \) only, since \( \partial \mathbf{C}/\partial z = 0 \) (see (9.2.8)). The isolines \( \mathbf{v}_a \mathbf{R} = \text{const.} \) are therefore parallel to the axis of rotation, and since \( \mathbf{v} \times \mathbf{v}_a \) is a tangent to these isolines (see (9.4.18)) we see that \( \mathbf{v} \times \mathbf{v}_a \) is normal to the planes \( z = \text{const.} \). On the other hand \( \mathbf{\Omega} \) is directed in the same direction, and we may then formulate the condition for stability in the form:

The autobarotropic, axisymmetric circulation is stable for displacements in the meridional plane if the rotation vector and the vorticity vector point in the same direction.

The same result could also have been obtained by noting that the flow is the same in all planes \( z = \text{const.} \) since the flow is autobarotropic. In any one of these planes we have a circular motion as indicated in Figure 9.9.
The vorticity vector for the two dimensional motion is perpendicular to the plane of the motion. The area of the hatched segment is

\[ \delta A = R \delta \lambda \delta R \] (9.4.21)

and the circulation around the area is

\[ \delta C = (R + \delta R) \delta \lambda (V + \frac{\partial V}{\partial R} \delta R) - R \delta A V \] (9.4.22)

The vorticity is

\[ \zeta_a = \lim_{\delta A \to 0} \frac{\delta C}{\delta A} = \delta A = \frac{\partial V}{\partial R} + \frac{V}{R} = \frac{1}{R} \frac{\partial VR}{\partial R} \] (9.4.23)

From (9.4.23) it is then easy to prove that

\[ \Omega_a = \frac{1}{8 \pi R^2} \frac{\partial C^2}{\partial R} \] (9.4.24)

The proof follows the same lines as the calculation leading to (9.4.20).

The results (9.4.19) or (9.4.24) determine the stability of an autobarotropic fluid in a clear form. Applied to the earth's atmosphere where \( \mathbf{\Omega} \), as discussed before, is pointing up from the North Pole, we will have a stable vortex if \( \zeta_a \) is positive.

The displacements in the meridional plane leading to the stability criterion are completely arbitrary, and they need not be in the radial direction. For the meteorologists, who wish to study the stability of the zonal currents as observed in the atmosphere, it is rather inconvenient to use \( R \) as the independent variable, since he will normally have his data plotted as a function of latitude. It is therefore convenient to transform the expression (9.4.23) using \( \zeta_a \). Let us consider the case of a zonal flow in a spherical surface parallel to the earth. In this case we have
$R = a \cos \varphi$

where $a$, the distance from the surface to the centre of the earth, is a constant. From (9.4.23) we find

$$\zeta_a = - \frac{1}{a \cos \varphi} \frac{1}{a \sin \varphi} \frac{\partial}{\partial \varphi} (\Omega a^2 \cos^2 \varphi) \quad (9.4.25)$$

or

$$\zeta_a = - \frac{1}{\sin \varphi} \frac{1}{\cos \varphi} \frac{\partial (\Omega \cos^2 \varphi)}{\partial \varphi} \quad (9.4.26)$$

$\zeta_a$ is the magnitude of the vorticity vector, which is normal to the surface $z = \text{const}$. In meteorological practice we normally use the vorticity component $\eta$ along the local vertical direction. We find

$$\eta = \zeta_a \sin \varphi = - \frac{1}{\cos \varphi} \frac{\partial (\Omega \cos^2 \varphi)}{\partial \varphi} \quad (9.4.27)$$

We divide $\Omega$ into $\Omega_E$ and $\Omega_r$. Noting further that $U_r = \Omega r \cos \varphi$, we find

$$\eta = 2\Omega \sin \varphi - \frac{1}{a \cos \varphi} \frac{\partial (U_r \cos \varphi)}{\partial \varphi} \quad (9.4.23)$$

The stability criterion for a horizontal, zonal current in the atmosphere is therefore that $\eta$, the vertical component of the absolute vorticity, is positive. The relative vorticity

$$\zeta_r = - \frac{1}{a \cos \varphi} \frac{\partial U_r \cos \varphi}{\partial \varphi} \quad (9.4.29)$$

is typically positive from the North Pole towards the lower latitudes to the centre of the subtropical jetstream. It is on the south side of the subtropical jetstream that one finds large negative values of $\zeta_r$, and because $f = 2\Omega \sin \varphi$ is becoming small at these latitudes it is possible that $\eta$ becomes very small or even slightly negative, which would indicate instability.

The stability criterion obtained here applies only for auto-barotropic fluids. We considered earlier, static stability questions where we investigated vertical displacements and found stability criteria. The general problem of finding the stability criteria for arbitrary displacements in a stratified, baroclinic fluid is most difficult and will not be considered here.
9.5 The Gradient Wind Vortex

The considerations in this chapter have so far been devoted to general considerations and an analysis of the circumpolar vortex. It is also possible to use considerations similar to those so far to other systems. We shall in this section consider the stability of the gradient wind vortex.

We consider a circular vortex where the pressure field is given. It is assumed that the wind everywhere can be determined from the gradient wind equation, i.e. there is a balance everywhere between the pressure force, the Coriolis force and the centrifugal force.

![Diagram](https://example.com/diagram.png)

**Figure 9.10**

The balance of forces in the cyclonic and anticyclonic cases (northern hemisphere) is shown in Figure 9.10 as analysed in the section on the gradient wind. The equation, expressing the balance of forces, and from which the gradient wind can be computed at each location is

\[ fV_G + \frac{V_G^2}{H} = fV_g \]  

(9.5.1)

where the two terms on the left side are the Coriolis force and the centrifugal force, respectively, while the right side of the equation is the pressure force, expressed through the use of the geostrophic wind. We emphasize that the pressure field is
given through a set of circular isobars, and that \( V \), therefore will vary as a function of \( R \), the distance from the centre. As shown in the analysis of the gradient wind we may also consider (9.5.1) as the equation for the anticyclonic case if we let \( R \) be negative in that case.

If we assume that the gradient wind vortex, illustrated in Figure 9.10, occurs in a horizontal surface where the specific volume is constant, we will have conservation of the circulation of the absolute wind. However, since we shall assume that \( f \) in (9.5.1) is a constant as we have done earlier in considerations of the gradient wind, we will have conservation of the circulation of the relative wind. For the particles located on a particular circle in Figure 9.10 we have

\[
S = \frac{V_R}{G} \tag{9.5.2}
\]

where \( S \), the spin, is the circulation except for a factor \( 2\pi \). If all the particles on a given circle are moved into a new circle we will have the same value of \( S \) for the displaced particles.

Let us now assume that the particles on the circle with radius \( R - dR \) are moved into the circle with radius \( R \). We shall denote the spin in the initial position by \( S - dS \). This value of the spin will be conserved under the displacement. When the particles arrive in the new position they will have a gradient wind \( V^* \) where

\[
V^*_G = \frac{S - dS}{R} \tag{9.5.3}
\]

In the new position the particles experience a pressure force which is exactly large enough to balance the Coriolis force and the centrifugal force at the distance \( R \). In the cyclonic case we have therefore a pressure force pointing towards the centre and equal to

\[
fV_G + \frac{V^2}{R}
\]

while we have the sum of the Coriolis force and the centrifugal force pointing away from the centre and equal to

\[
fV^*_G + \frac{V^*}{R}
\]

since these forces are determined by the speed \( V^*_G \) of the particles.
The condition for stability is naturally that the net force is pointing towards the centre, since the particles were displaced in the direction away from the centre. Thus, the stability condition is
\[ fV_G^* - \frac{V_G^2}{R} < fV_G + \frac{V_G^2}{R} \]

We have the expression (9.5.3) for \( V_G^* \), while \( V_G = S/R \). Inserting these expressions in (9.5.3) we find after evaluation
\[ -\frac{f}{R} \Delta S - 2 \frac{S}{R^2} \Delta S + \frac{1}{R^3} (\Delta S)^2 < 0 \]  
(9.5.4)

Disregarding the last term in (9.5.4) because it is small, of the second order, we may write (9.5.4), the condition for stability, in the form
\[ \frac{dS}{R} \left( f + 2 \frac{S}{R^2} \right) > 0 \]  
(9.5.5)

Dividing by \( dR \) and noting that the vorticity is (see (9.4.23))
\[ \zeta = \frac{1}{R} \frac{dS}{dR} \]  
(9.5.6)

we find
\[ \zeta \left( f + 2 \frac{V_G}{R} \right) > 0 \]  
(9.5.7)

It is thus obvious from (9.5.7) that a cyclonic vortex is always stable because \( \zeta > 0 \) and \( f + 2 \frac{V_G}{R} < 0 \).

The conditions for the anticyclonic vortex may be obtained by repeating the arguments or simply by replacing \( \zeta \) and \( R \) by \( -\zeta \) and \( -R \), respectively. As the stability condition for the anticyclonic case we therefore get:
\[ -\zeta \left( f - 2 \frac{V_G}{R} \right) > 0 \]  
(9.5.8)
or
\[ \zeta \left( f - 2 \frac{V_G}{R} \right) < 0 \]  
(9.5.9)

Since \( \zeta < 0 \), we find that the stability condition is
\[ f - 2 \frac{V_G}{R} > 0 \]
or
\[ V_G < \frac{1}{2} fR \]  
(9.5.10)
In the anticyclonic case there is therefore a possibility for instability, which will occur if $V_G$ is too large. The limiting case of neutral conditions is found when

$$V_G = \frac{1}{2} f R$$  \hspace{1cm} (9.5.11)$$

We find from (9.5.1) (remembering to replace $R$ by $-R$) that $V = \frac{1}{4} f R = \frac{1}{2} V_G$ or $V_G = 2V$ in the limiting case, but a comparison with the analysis of the gradient wind (see section 8.4) shows that this coincides with the case when the gradient wind balance is impossible.

The stability condition (9.5.10) may also be expressed in the form

$$\omega < \frac{1}{2} f$$  \hspace{1cm} (9.5.12)$$

where $\omega$ is the local angular velocity, determined by the relation $V_G = \omega R$. In this form the stability condition says simply that the angular velocity in the gradient wind vortex has an upper limit in the anticyclonic case.
Chapter X

DISCONTINUITY SURFACES

10.1 Introduction

The system which has been considered so far has been characterized by a continuous variation of the dependent variables. It has been tacitly understood that the three-dimensional wind, pressure, density and temperature vary continuously within the atmosphere. It is an observed fact that developments in the atmosphere are occasionally such that very strong gradients develop. The atmospheric fronts (described in Synoptic Meteorology (Volume II)) are examples of systems in which very strong temperature gradients exist. As another example we may mention the tropopause which is characterized by an abrupt change in the temperature lapse rate. Still another example is an inversion in the lower part of the atmosphere. An inversion is characterized as a surface of maximum temperature. It has therefore also an abrupt change in the vertical component of the temperature gradient.

Many of the systems mentioned above can be treated as true discontinuities although such treatments always represent idealizations. We recall in this connexion the boundary condition which applies at a discontinuity surface. The dynamic boundary condition is that the pressure is continuous across the surface. If the pressure gradient is discontinuous across the surface, we call the surface a discontinuity surface of zero order. A discontinuity surface of the first order is one where the pressure and its first derivative are continuous, but the second derivative is discontinuous. It is obvious that we can define discontinuity surfaces of any order, but only the two mentioned above will be of interest here. Let us furthermore note that the dynamic boundary condition mentioned above applies in a flow with no friction.

We shall in this chapter restrict ourselves to a consideration of the idealized cases which can be treated using the dynamic boundary condition mentioned above. It should be noted, however, that such a treatment is incomplete since we do not consider the kinematic boundary condition which determines the motion of the discontinuity surface. In addition, we assume here that the surface of discontinuity already exists, and we are therefore not considering the important question of how these surfaces are created by the dynamical and kinematical conditions in the atmosphere.
10.2 Discontinuity of the Zero'th Order

Figure 10.1 shows the idealized situation which we shall consider here. The discontinuity surface has the equation

$$z = f(x, y, t)$$  \hspace{1cm} (10.2.1)

giving the height $z$ above the ground as a function of the horizontal co-ordinates and time.

![Figure 10.1](image)

The dynamic boundary condition is that the pressures at $A_1$ and $A_2$ are the same, just as the pressure at $B_1$ is equal to the pressure at $B_2$. If $(\delta p)_2 = p(B_2) - p(A_2)$ and $(\delta p)_1 = p(B_1) - p(A_1)$ we can express the dynamic boundary condition in the form

$$(\delta p)_2 = (\delta p)_1$$  \hspace{1cm} (10.2.2)

We consider now the vector

$$\delta \mathbf{r} = \frac{A_2B_2}{A_1B_1} = \frac{A_2B_2}{A_1B_1}$$  \hspace{1cm} (10.2.3)

where the latter relation holds since $A_1$ is infinitely close to $A_2$, and $B_1$ infinitely close to $B_2$. Dividing $\delta \mathbf{r}$ in its horizontal and vertical components we have

$$\delta \mathbf{r} = \delta \mathbf{r}_h + \delta \mathbf{r}_v$$  \hspace{1cm} (10.2.4)

where $\delta \mathbf{r}_h = A_1C_1$ and $\delta \mathbf{r}_v = C_1B_1$, where $\mathbf{k}$ is a vertical unit vector.

Let us now consider the surface of discontinuity. The height is given by (10.2.1). We may characterize the surface by a set of isolines of constant $z$ in a horizontal surface in the same way as we describe an isobaric surface by its isolines of constant height. The quantity $\delta z$ in Figure 10.1 is the height difference for the points $A$ and $B$. We have therefore
\[ \delta z = \frac{\partial \delta z}{\partial x} \delta x + \frac{\partial \delta z}{\partial y} \delta y = \mathbf{v}_D^z \cdot \delta \mathbf{r}_h \]  

where \( \mathbf{v}_D^z \) is the gradient of the field \( z \) for the discontinuity surface. \( \mathbf{v}_D^z \) is therefore a horizontal vector. In view of (10.2.5) we may write (10.2.4) in the form

\[ \delta \mathbf{r} = \delta \mathbf{r}_h + (\mathbf{v}_D^z \cdot \delta \mathbf{r}_h) \mathbf{k} \]  

(10.2.6)

Applying (10.2.2) we have

\[ \mathbf{v}_P_1 \cdot \delta \mathbf{r} = \mathbf{v}_P_2 \cdot \delta \mathbf{r} \]  

(10.2.7)

or

\[ \mathbf{v}_h P_1 \cdot \delta \mathbf{r}_h + \frac{\partial \mathbf{v}_P_1}{\partial z} \delta z = \mathbf{v}_h P_2 \cdot \delta \mathbf{r}_h + \frac{\partial \mathbf{v}_P_2}{\partial z} \delta z \]  

(10.2.8)

Substituting from (10.2.5) in (10.2.8) we get

\[ \left( \mathbf{v}_h P_1 + \frac{\partial \mathbf{v}_P_1}{\partial z} \mathbf{v}_D^z \right) \cdot \delta \mathbf{r}_h = \left( \mathbf{v}_h P_2 + \frac{\partial \mathbf{v}_P_2}{\partial z} \mathbf{v}_D^z \right) \cdot \delta \mathbf{r}_h \]  

(10.2.9)

Since (10.2.9) must apply for all vectors \( \delta \mathbf{r}_h \) we have from (10.2.9)

\[ \mathbf{v}_D^z = - \frac{\mathbf{v}_h P_1 - \mathbf{v}_h P_2}{\frac{\partial \mathbf{v}_P_1}{\partial z} - \frac{\partial \mathbf{v}_P_2}{\partial z}} \]  

(10.2.10)

which is the equation for the slope of the discontinuity surface, also known as Margules' equation.

It is easy to write (10.2.10) in other forms. We first of all express the horizontal pressure gradients in terms of the geostrophic windspeeds

\[ \mathbf{v}_h P = - \frac{f}{\mathbf{k}} \times \mathbf{v}_g \]  

(10.2.11)

If we assume hydrostatic equilibrium we have

\[ \mathbf{v}_D^z = - \frac{f}{g} \frac{\mathbf{k} \times (\rho_1 \mathbf{v}_D^1 - \rho_2 \mathbf{v}_D^2)}{\rho_1 - \rho_2} \]  

(10.2.12)

In order to investigate the implications of (10.2.12) we introduce a local co-ordinate system where the \( x \)-axis is along the direction of \( \mathbf{v}_D^z \) while the \( y \)-axis is normal to the \( x \)-axis in such a way that a normal co-ordinate system is formed. We get then

\[ \frac{\partial z}{\partial x} = \frac{f}{g} \frac{\rho_1 \mathbf{v}_D^1 - \rho_2 \mathbf{v}_D^2}{\rho_1 - \rho_2} \]  

(10.2.13)
Since \( \frac{\partial z}{\partial x} > 0 \) by definition, and \( \rho_1 > \rho_2 \) for stability of the system, we have

\[
\rho_1 v_1 > \rho_2 v_2
\]  

(10.2.14)

We get further from (10.2.12) that

\[
\frac{\partial z}{\partial y} = -\frac{f}{g} \frac{\rho_1 u_1 - \rho_2 u_2}{\rho_1 - \rho_2} = 0
\]

(10.2.15)

since the \( y \)-axis is defined in such a way that \( \frac{\partial z}{\partial y} = 0 \). It follows therefore that

\[
\rho_1 u_1 = \rho_2 u_2
\]

(10.2.16)

Considering (10.2.14) and (10.2.16) we may distinguish three different cases:

(a) \( u_1 \) (and \( u_2 \)) positive

We have in this case wind directions which are such that the flow normal to the intersection of the discontinuity with the horizontal surface is from the less dense (warmer) to the more dense (colder) side of the intersection.

![Figure 10.2](image)

This situation is illustrated in Figure 10.2, where \( II \) is the intersection of the discontinuity surface with the horizontal surface. Let \( \overrightarrow{OP_1} = \rho_1 \overrightarrow{v_1} \) be the vector in the denser airmass. The vector \( \overrightarrow{OP_2} = \rho_2 \overrightarrow{v_2} \) must then be such that the point \( P_2 \)
is located anywhere on a line through $P_1$ parallel to II because (10.2.16) must be satisfied. However since (10.2.14) must also be satisfied we can only use positions $P_2$ which have $y$-co-ordinates smaller than the $y$-co-ordinates for $P_1$. The vector $\overrightarrow{OP_2}$ shown in Figure 10.2 is therefore only one of many possibilities for $\overrightarrow{P_2V_2}$. Knowing the directions $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ we can draw the isobars (indicated by the dashed lines) in the two airmasses. Because of (10.2.14) we must always have a cyclonic turning of the vector $\overrightarrow{V}$ when we go from the air mass with density $\rho_2$ to the other air mass.

(b) $u_1$ (and $u_2$) equal to zero

The flow is, in this case, parallel to II in Figure 10.3. If we again start from $\overrightarrow{P_1V_1} = \overrightarrow{OP_1}$, we can have all vectors $\overrightarrow{P_2V_2} = \overrightarrow{OP_2}$ where $P_2$ can be located at any point along II as long as $P_2$ is on the same side of $P_1$ as 0.

(c) $u_1$ (and $u_2$) negative

The flow is illustrated in Figure 10.4.

If $\overrightarrow{OP_1} = \overrightarrow{P_1V_1}$ is the vector in the denser air mass, then the point $P_2$ must be located on a line through $P_1$ parallel to II as indicated on Figure 10.4. The isobars (dashed lines) are drawn parallel to the vector $\overrightarrow{P_1V_1}$ in the air mass with density $\rho_1$ and parallel to $\overrightarrow{P_2V_2}$ in the other air mass. We notice again the cyclonic turning of the wind as we cross the intersection as indicated by the directions of the vectors $\overrightarrow{P_1V_2}$ and $\overrightarrow{P_2V_2}$. 
The discontinuity surfaces are called frontal surfaces in the atmospheric case, and the intersections II are called fronts. The first case is normally called a warm front because the wind blows from the warmer towards the colder airmass. The third case is then the cold front case.

10.3 Discontinuity Surfaces of the First Order

For these surfaces we find that the expression (10.2.10) for the slope becomes ill-defined because the pressure gradient is continuous across the surface. In order to find the slope of the discontinuity surface of the first type we note from hydrostatic considerations that the density is continuous across the surface, since $\partial p/\partial z$ is the same on both sides. However, from the gas equation it follows also that the temperature is continuous across the surface. We can use any of these conditions to determine the slope, but since temperature is the most common parameter available in practice we shall use it here.

Corresponding to (10.2.7) we get in this case

$$\nabla T_1 \cdot \delta \mathbf{r} = \nabla T_2 \cdot \delta \mathbf{r}$$

(10.3.1)
or
\[
\nabla T_1 \cdot \delta \mathbf{r}_h + \frac{\partial T_1}{\partial z} (v_{Dz} \cdot \delta \mathbf{r}_h) = \nabla T_2 \cdot \delta \mathbf{r}_h + \frac{\partial T_2}{\partial z} (v_{Dz} \cdot \delta \mathbf{r}_h)
\]
(10.3.2)

where we have used (10.2.5). From (10.3.2) we obtain solving for \(v_{Dz}\)

\[
v_{Dz} = -\frac{\nabla T_1 - \nabla T_2}{\frac{\partial T_1}{\partial z} - \frac{\partial T_2}{\partial z}}
\]
(10.3.3)

which is one expression for the slope of a discontinuity surface of the first order.

We may naturally introduce the lapse rate \(\gamma = -\partial T/\partial z\), and (10.3.3) becomes

\[
v_{Dz} = \frac{\nabla T_1 - \nabla k T_2}{\gamma_1 - \gamma_2}
\]
(10.3.4)

Formula (10.3.4) can be used to calculate the slope of the tropopause, which is characterized by a change in lapse rate. If subscript 1 corresponds to the troposphere and 2 to the stratosphere, we have in general \(\gamma_1 > 0\) and \(\gamma_1 - \gamma_2 > 0\), because the conditions in the lower stratosphere are much more stable than in the troposphere.

Introducing a coordinate system analogous to the one introduced in section 10.2 we find

\[
\frac{\partial z}{\partial x} = \frac{\frac{\partial z}{\partial T_1}}{\frac{\partial z}{\partial T_2}}
\]
(10.3.5)

and

\[
0 = \frac{\partial T_1}{\partial y} - \frac{\partial T_2}{\partial y}
\]
(10.3.6)

Since \((\partial z/\partial x) > 0\) we get

\[
\frac{\partial T_1}{\partial z} > \frac{\partial T_2}{\partial z}
\]
(10.3.7)

which says that the temperature gradient normal to the tropopause intersection with a horizontal surface is larger in the troposphere than in the stratosphere. The typical temperature distribution around the tropopause is shown in Figure 10.5, where it has been assumed that \(\gamma_2 = 0\), corresponding to isothermal conditions in the lower stratosphere.

The case illustrated in Figure 10.5 is such that \((\partial T_1/\partial x) > 0\) and \((\partial T_2/\partial x) < 0\) in which case (10.3.7) is obviously satisfied. If we follow the line AB, located on a constant level chart, we find that the tropopause intersection with the horizontal surface is located at a temperature minimum.
Figure 10.5
Chapter XI

ATMOSPHERIC WAVES

11.1 Introduction

The concept of a wave is well known from general experience (water waves, sound waves). We shall in this section consider the various wave phenomena which are found in the atmosphere. The equations derived in the earlier chapters of this compendium are quite general since they are derived from basic physical laws (Newton's second law, first law of thermodynamics, etc.) and adapted to a rotating planet surrounded by an atmosphere. It is therefore understandable that they are capable of describing many kinds of wave phenomena some of which are of paramount importance for the understanding of the large-scale flow in the atmosphere although others have little relation to this flow.

We shall in this chapter try to isolate various waves in a precise form in order to distinguish clearly between them. In an effort to keep the treatment as simple as possible we will restrict our attention to sinusoidal waves. Since it is impossible to obtain very general solutions for the unmodified equations because of their mathematical complexity, it will also be necessary to obtain solutions under special conditions. Here we shall make use of the so-called perturbation method which is described in detail in section 11.3.

11.2 Sinusoidal Waves

We consider a dependent variable \( h = h(x, y, z, t) \). Let us assume that \( h \) has the form

\[
h = A(y, z) \cos(k(x-ct)) \tag{11.2.1}
\]

\( A(y, z) \) is called the amplitude. It is in general a function of \( y \) and \( z \), but in many applications it is assumed that \( A \) is a function of one of the variables only or is simply a constant. The wave number \( k \) is related to the wavelength \( L \) by the relation \( k = 2\pi/L \), while the frequency \( \nu = kc \) is related to the period \( T \) by the relation \( \nu = 2\pi/T \). The expression (11.2.1) describes a simple sinusoidal wave form which propagates in the \( x \)-direction with the wave speed \( c \) as described in section 6.14.
In theoretical considerations it is very often convenient to write the expression for the wave in the form

\[ \text{Re} \left[ B(y,z) e^{i k(x-ct)} \right] \]  

(11.2.2)

where \( \text{Re} \) means the real part of the complex number in the bracket. We note first of all that (11.2.2) is equivalent to (11.2.1) if \( c \) is real, because

\[ \text{Re} \left[ B e^{i k(x-ct)} \right] = B_r \cos k(x-ct) - B_i \sin k(x-ct) \]  

(11.2.3)

when \( B = B_r + i B_i \). (11.2.3) may also be written

\[ B_r \cos k(x-ct) - B_i \sin k(x-ct) = B_A \cos [k(x-ct) + \delta] \]  

(11.2.4)

where

\[ B_A^2 = B_r^2 + B_i^2 \]  

(11.2.5)

and

\[ \tan \delta = \frac{B_i}{B_r} \]  

(11.2.6)

It is thus seen that (11.2.4) is exactly the same as (11.2.1) except for for a different position of the maximum.

(11.2.2) is on the other hand a generalization of (11.2.1) if \( c \) is complex. Let \( c = c_r + i c_i \). We then have

\[ \text{Re} \left[ B e^{i k(x-ct)} \right] = e^{k c_i t} \text{Re} \left[ B e^{i k(x-c_r t)} \right] \]  

(11.2.7)

The last term in (11.2.7) is of the same form as (11.2.2) when \( c \) is real. The first term in (11.2.7) may be considered as part of a time dependent amplitude. We obtain

\[ e^{k c_i t} \text{Re} \left[ B e^{i k(x-c_r t)} \right] = e^{k c_i t} B_A \cos [k(x-c_r t) + \delta] \]  

(11.2.8)

according to (11.2.4). If we denote

\[ B_A^* = B_A e^{k c_i t} \]  

(11.2.9)
we may say that \( B^*_A \) increases with time if \( c_1 > 0 \), is constant if \( c_1 = 0 \), and decreases if \( c_1 < 0 \).

In the analysis of atmospheric phenomena, of which we shall see some examples in this chapter, it is normally the problem to find the values (real or complex) which the wave speed \( c \) must have in order that (11.2.2) may be a solution to the (approximate) equations describing the phenomena under consideration. When such a solution has been found we can classify it according to the rules

\[
\begin{align*}
    c_1 > 0, & \quad \text{unstable} \\
    c_1 = 0, & \quad \text{neutral} \\
    c_1 < 0, & \quad \text{stable}.
\end{align*}
\]

11.3 The Perturbation Method

The general meteorological equations (developed in Chapters I and II and collected in Chapter III) are very complicated from a mathematical point of view. The set of equations belongs to the group called: a closed set of coupled, non-linear, partial differential equations. The mathematical theory for these equations is not developed in general terms and it is in only a few special cases that we know solutions to such equations. Realizing this situation we have two possibilities open to us: the first is to obtain particular solutions using numerical methods, while the second is to reduce the equations in such a way that they become mathematically tractable. Both methods are used extensively in meteorological research. We shall in this section describe one method, the so-called perturbation method, which may be used to obtain solutions to a set of equations which can be handled from a mathematical point of view. The main purpose of the perturbation method is to reduce the non-linear equations to a set of linear equations. This can naturally only be done under simplifying assumptions.

We may describe the general aspects of the perturbation method as follows:

(a) We find a (relatively simple) solution to the set of equations governing the physical problems at hand. The state of the atmosphere corresponding to this solution is called the basic state and is denoted by the symbols: \( \bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{\rho}, \bar{T} \).
(b) We consider a new state consisting of the basic state plus an additional field, called the perturbation field. Denoting this field by primed symbols we get: \( \tilde{u} + u', \, \tilde{v} + v', \, \tilde{w} + w', \, \tilde{p} + p', \, \tilde{\rho} + p', \, \tilde{T} + T' \). It is now required that the total field \((\tilde{\bar{u}} + u', \text{ etc.})\) shall be a solution to the problem.

(c) Substituting the total field \((\tilde{\bar{u}} + u', \text{ etc.})\) into the equations valid for the problem we make the major assumption in the perturbation method that the perturbation is small. In other words we shall neglect all quantities which are of the second or higher orders in the perturbation quantities. It is this assumption which reduces the non-linear equations to a set of linear partial differential equations, which in certain cases can be solved mathematically.

(d) Solutions to the linear equations, describing the behaviour of the perturbation fields, are sought in terms of the wave type solutions described in section 11.2.

(e) From the solution we may determine if the basic state is unstable \((c_1 > 0)\), neutral \((c_1 = 0)\) or stable \((c_1 < 0)\). We may in addition determine the speed \((c_r)\) of the wave and the structure of the perturbation wave.

In the following sections we shall consider several examples of the general procedure outlined above.

11.4 **Sound Waves**

It is a general experience that sound waves can propagate through the atmosphere. We shall, here, find the speed of propagation of sound waves. In order to get the sound waves in a pure form we shall consider a particularly simple case.

(a) Let the basic state consist of an airmass in which we may assume that \( \tilde{\bar{\sigma}} \) is a constant (homogeneous atmosphere), and in which there is no motion, i.e. \( \tilde{u} = \tilde{v} = \tilde{w} = 0 \). The basic state is therefore described by the equation

\[
\tilde{\rho} = \text{const.}
\]  

(11.4.1)

when we disregard the effects of gravity.

(b) The total field is: \( u = u', \, v = v', \, w = w', \, p = \tilde{\rho} + p' \), and \( \bar{\sigma} = \tilde{\bar{\sigma}} + \bar{\sigma}' \).
(c) In describing the sound waves we shall assume that we can neglect rotation \((f = 0)\), heating \((\mathcal{H} = 0)\) and friction \((\mathcal{P} = 0)\). The equations describing the phenomenon are

\[
\begin{align*}
\frac{du}{dt} &= -c \frac{dp}{dx} \\
\frac{dv}{dt} &= -c \frac{dp}{dy} \\
\frac{dw}{dt} &= -c \frac{dp}{dz} \\
\frac{dp}{dt} &= -\gamma p \nu \cdot \nu; \quad \gamma = \frac{c_p}{c_v}
\end{align*}
\]  

(11.4.2)

where the last equation is a form of the thermodynamic equation. Inserting the total fields in (11.4.2) and neglecting higher order terms we get

\[
\begin{align*}
\frac{du'}{dt} &= -a \frac{dp'}{dx} \\
\frac{dv'}{dt} &= -a \frac{dp'}{dy} \\
\frac{dw'}{dt} &= -a \frac{dp'}{dz} \\
\frac{dp'}{dt} &= -\gamma p \left( \frac{2u'}{2x} + \frac{2v'}{2y} + \frac{2w'}{2z} \right)
\end{align*}
\]  

(11.4.3)

We may now eliminate \(u'\), \(v'\), and \(w'\) from the system (11.4.3) by differentiating the last equation with respect to time and substituting from the first three equations. We get then the famous wave-equation

\[
\frac{\partial^2 p'}{\partial t^2} = \gamma a \frac{\partial^2 p'}{\partial x^2}
\]  

(11.4.4)

where

\[
\nu^2 p' = \frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 p'}{\partial z^2}
\]  

(11.4.5)
Let us now consider what happens if by some means we generate a decrease of the pressure at some point 0 with no change in pressure elsewhere (see Figure 11.1).

![Figure 11.1]

It can be seen from the equations of motion (11.4.3) that everywhere the particles will be accelerated towards 0. Thus $\frac{\partial u'}{\partial t} < 0$ at points on the positive part of $x$-axis, while $\frac{\partial u'}{\partial t} > 0$ on the negative part of the same axis. Similar arguments show that an analogous behaviour will take place along the $y$- and $z$-axes. A short time later we will then have $\nabla \cdot \mathbf{v}' < 0$ (convergence), and $\frac{\partial p'}{\partial t} > 0$ at 0, such that the pressure will move back towards $\bar{p}$. This process will continue until the pressure at 0 is larger than $\bar{p}$ in 0, i.e. $p' > 0$. From the equations of motion we find that we will get $\nabla \cdot \mathbf{v}' > 0$ (divergence) which according to the last equation in (11.4.3) will result in $\frac{\partial p'}{\partial t} < 0$, and the pressure will start to decrease again at 0.

The net result as far as the point 0 is concerned is therefore that the pressure will undergo oscillation, and the particles around 0 will participate in motions around their equilibrium positions resulting in alternating fields of divergence and convergence. It is however also obvious that all these oscillations of pressure and velocity components will propagate through the medium. We shall now find the speed of propagation. Since we have complete symmetry in the $x$-, $y$- and $z$- directions, we may restrict ourselves to a consideration of the distance $r$ from the centre. Then, replacing (11.4.4), we have:
\[ \frac{\partial^2 p'}{\partial t^2} = \gamma \frac{\partial}{\partial r} \left( \frac{\partial p'}{\partial r} \right) \]  

(11.4.6)

Assuming now that

\[ p' = \dot{p} \cos k(r-ct) \]  

(11.4.7)

we find by substitution in (11.4.6) that

\[ -k^2 \frac{\partial^2 p}{\partial r^2} \cos k(r-ct) = -\gamma \frac{\partial}{\partial r} p \frac{\partial}{\partial r} \cos k(r-ct) \]  

(11.4.8)

which leads to

\[ c = \pm \sqrt{\frac{\gamma \rho}{p}} \]  

(11.4.9)

(11.4.9) gives the speed of propagation of sound waves, originally derived by Laplace in this form. It may also be written in the following form using the gas equation

\[ c = \pm \sqrt{\frac{\gamma R T}{p}} \]  

(11.4.10)

For \( c_p = 1004 \, m^2 \, s^{-2} \, \text{deg}^{-1} \), \( c_v = 717 \, m^2 \, s^{-2} \, \text{deg}^{-1} \), \( R = 287 \, m^2 \, s^{-2} \, \text{deg}^{-1} \) and \( T = 273^\circ K \)

we find \( c = 331 \, m \, s^{-1} \) which may then be considered as the speed of sound in an isothermal atmosphere. The sound waves propagate in the atmosphere through a series of compressions and expansions, and they will therefore only propagate if the medium is compressible. It is obvious that the derivation of the speed of sound will be very complicated if we use a more realistic basic state, but it is beyond the scope of this compendium to treat these cases.

11.5 Gravity Waves

The purpose of this section is to consider the waves which will exist in the atmosphere mainly because the atmosphere is in the field of gravity. In order to study these waves in a pure form, we shall neglect the rotation of the earth, but assume hydrostatic conditions. It is very cumbersome to obtain solutions for the gravity waves in any, but the most simple situations. Partly because of this situation, and partly as a preparation for later studies of large scale atmospheric motion we shall introduce a simple two-level model of the atmosphere. Such models are widely used in studies of atmospheric flow, and the main principle, in the construction of the model,
is that vertical derivatives are replaced by finite differences. Such a procedure becomes more accurate the smaller the interval over which the finite difference is taken. It is therefore necessary to divide the atmosphere using many levels in the vertical direction, if we want to reduce the error, the so-called truncation error, committed by replacing the derivative by a finite difference. However, experience shows that the main effects, although containing truncation errors, are included in even the minimal division in the vertical direction.

The basic state shall be a state of rest, i.e. \( \vec{u} - \vec{v} = \omega = 0 \). The geopotential of the isobaric surfaces can in this situation be considered as constant for each isobaric surface, i.e. \( \dot{\phi} = \dot{\phi}(p) \). In order to keep the analysis simple we shall disregard all variations in the meridional direction. It follows that all \( \dot{\gamma} = 0 \) and, in particular, that \( \nu' = 0 \). We shall in addition assume that the motion is adiabatic and without friction. The thermodynamic equation is under the first assumption that

\[
\frac{d \ln \theta}{dt} = 0 \quad (11.5.1)
\]

where \( \theta \) is the potential temperature.

It is convenient for later applications to express (11.5.1) in terms of the geopotential. Since we have the hydrostatic relation that

\[
\frac{d}{dp} = - \frac{\rho}{\rho_0}
\]

we shall first express (11.5.1) in terms of the specific volume \( \rho \). We have

\[
\theta = T \left( \frac{R}{c_p} \right) \frac{C_v}{R} = \frac{R}{p_0} \left( \frac{p}{p_0} \right) \frac{C_v}{R} = \frac{R}{\rho} \frac{C_v}{R} \frac{R}{\rho} \quad (11.5.3)
\]

and thus

\[
\ln \theta = \ln \rho + \frac{C_v}{\rho} \ln p + \text{const.} \quad (11.5.4)
\]

Because we shall use the pressure system here we have

\[
\frac{\Delta \ln \theta}{\Delta t} = \frac{\partial \ln \rho}{\partial t}; \quad \nabla \ln \theta = \nabla \ln \rho
\]

and (11.5.1) becomes

\[
\frac{1}{a} \frac{\partial a}{\partial t} + \frac{1}{a} \vec{v} \cdot \nabla a + \omega \frac{\partial \ln \theta}{\partial \mathbb{p}} = 0
\] (11.5.6)

or, using (11.5.2)

\[
\frac{d}{dt} \left( \frac{\partial \phi}{\partial \mathbb{p}} \right) + \nabla \cdot \left( \nabla \frac{\partial \phi}{\partial \mathbb{p}} \right) + \sigma \omega = 0
\] (11.5.7)

where

\[
\sigma = -a \frac{\partial \ln \theta}{\partial \mathbb{p}}
\] (11.5.8)

It is naturally also possible to express \( \sigma \) in terms of the geopotential \( \phi \). We get from (11.5.4):

\[
\frac{\partial \ln \theta}{\partial \mathbb{p}} = \frac{1}{a} \frac{\partial a}{\partial \mathbb{p}} + \frac{c_v}{c_p} \frac{1}{\mathbb{p}}
\] (11.5.9)

and

\[
\sigma = -\left[ \frac{\partial a}{\partial \mathbb{p}} + \frac{c_v}{c_p} \frac{a}{\mathbb{p}} \right]
\] (11.5.10)

which may also be written

\[
\sigma = \frac{\partial^2 \phi}{\partial \mathbb{p}^2} + \frac{c_v}{c_p} \frac{1}{\mathbb{p}} \frac{\partial \phi}{\partial \mathbb{p}}
\] (11.5.11)

It follows from the initial remark that \( \tilde{\phi} = \tilde{\phi}(\mathbb{p}) \) in the basic state that there exists a function \( \tilde{\sigma} = \tilde{\sigma}(\mathbb{p}) \), evaluated from (11.5.11) using \( \tilde{\phi} = \tilde{\phi}(\mathbb{p}) \), characterizing the stability of the basic state. We shall assume \( \tilde{\sigma} > 0 \) such that we have static stability.

The two-level model is now constructed in the following way (see Figure 11.2). The atmosphere is divided into two equal parts with respect to mass. Assuming that the surface pressure is approximately 1000 mb, we select the 500 mb pressure surface as the division between the upper and the lower parts. The upper part of the atmosphere will be represented by the level 250 mb (denoted by subscript 1), while the lower part of the atmosphere will be represented by the level 750 mb (denoted by subscript 3).
The division between the upper and the lower part is the 500 mb surface (denoted by subscript 2), while the upper boundary is at $p = 0$ (denoted by subscript 0) and the lower boundary at $p = 1000$ mb (denoted by subscript 4).

The equations of motion for the $u$-component in the upper and lower layers are, respectively:

\[
\frac{\partial u'_1}{\partial t} = - \frac{\partial \psi'_1}{\partial x} \quad (11.5.12)
\]

\[
\frac{\partial u'_3}{\partial t} = - \frac{\partial \psi'_3}{\partial x} \quad (11.5.13)
\]

The continuity equations, applied to the upper and lower layers, respectively, are

\[
\omega'_2 = - P \frac{\partial u'_1}{\partial x} \quad (11.5.14)
\]

\[
\omega'_4 - \omega'_2 = - P \frac{\partial u'_3}{\partial x} \quad (11.5.15)
\]

where $P = 500$ mb, and where we have taken finite differences across half the mass of the atmosphere. We note again that (11.5.14) and (11.5.15) are simple because $\psi'_{12} = 0$ in the evaluation by assumption, and $\omega'_0 = 0$ (the upper boundary condition).

The thermodynamic equation (11.5.7) applied at level 2 becomes

\[
- \frac{1}{P} \frac{\partial (\psi'_1 - \psi'_3)}{\partial t} + \overline{\sigma}_2 \omega'_2 = 0 \quad (11.5.16)
\]
where again we have taken a finite difference across half of the atmosphere, but this time from level 1 to level 3. Considering the system \((11.5.12) - (11.5.16)\) we count 6 variables \((u'_{1}, u'_{3}, \phi'_{1}, \phi'_{3}, \omega'_{2} \text{ and } \omega'_{4})\), but only 5 equations. The missing equation comes from the lower boundary condition. Assuming that the earth is flat we have \(w = 0\) at \(z = 0\) as the lower boundary condition. It is not straightforward to apply such a condition in a system with pressure as the vertical co-ordinate. We have, however, that the lower boundary is at 1000 mb, approximately. As a reasonable approximation we shall apply the condition \(w = 0\) at \(p = p_{4}\). We have then as our final equation

\[
\frac{d\phi'_{4}}{dt} - \frac{d}{dt} \left( \frac{\phi'_{4}}{\partial} \right) = \omega'_{4} \left( \frac{\phi'}{\partial p} \right) = 0 \tag{11.5.17}
\]

(11.5.17) does not close the system because \(\phi'_{4}\) has been introduced. This is due to the finite differences, and we shall close the system by expressing \(\phi'_{4}\) in terms of \(\phi'_{1}\) and \(\phi'_{3}\). It is customary to obtain \(\phi'_{4}\) by simple linear extrapolation, i.e.

\[
\phi'_{4} = \frac{3}{2} \phi'_{3} - \frac{1}{2} \phi'_{1} \tag{11.5.18}
\]

The linear perturbation equations are the equations \((11.5.12)\) to \((11.5.18)\). We introduce for each of the variables a sinusoidal wave of the form

\[
(\quad)' = (\quad) e^{ik(x-ct)} \tag{11.5.19}
\]

where \((\quad)'\) denotes any of the variables and \((\quad)\) its amplitude. We then get from these equations:

\[
- \text{i}k \hat{a}_{1} = - \text{i}k \hat{\phi}_{1}
\]

\[
- \text{i}k \hat{a}_{3} = - \text{i}k \hat{\phi}_{3}
\]

\[
\hat{\omega}_{2} = - \text{P i}k \hat{a}_{1}
\]

\[
\hat{\omega}_{4} = - \text{P i}k \hat{a}_{3}
\]

\[
\frac{\text{i}k}{\text{P}} \left( \hat{\phi}_{1} - \hat{\phi}_{3} \right) + \overline{\sigma}_{2} \hat{\omega}_{2} = 0
\]

\[
\omega_{4} = + \text{P i}k \left( \frac{1}{2} \hat{\phi}_{1} - \frac{3}{2} \hat{\phi}_{3} \right)
\]
It is a straightforward matter to eliminate all variables, but two, from the system (11.5.20). If we retain the variables $\alpha_1$ and $\alpha_3$, we get

\[
(c^2 - \sigma_2^2 p^2) \alpha_1 - c^2 \alpha_3 = 0 \tag{11.5.21}
\]

\[
(P + \frac{1}{2} \frac{\rho c^2}{4}) \alpha_1 + (P - \frac{3}{2} \frac{\rho c^2}{4}) \alpha_3 = 0
\]

The system (11.5.21) consists of two homogeneous linear equations in the variables $\alpha_1$ and $\alpha_3$. From the theory for linear homogeneous equations we know that the condition for non-trivial solutions to such equations is that the determinant is zero. We shall, however, consider two special cases of (11.5.21).

Suppose first that $\sigma_2 = 0$. This condition corresponds, according to (11.5.8), to an adiabatic stratification, i.e. $\bar{\vartheta} = \text{const}$. From the first equation in (11.5.21) we get in this case $\alpha_1 = \alpha_3$, which means, among other things, that there is as much divergence in the upper part as in the lower part of the atmosphere. The second equation becomes

\[
(2P - \frac{\rho c^2}{4}) \alpha_1 = 0 \tag{11.5.22}
\]

from which it follows that

\[
c = \pm \sqrt{2P - \frac{\rho c^2}{4}} = \pm \sqrt{\frac{P_4}{\rho}}
\]

The waves described by (11.5.23) are due entirely to conditions at the lower boundary. They are called external gravity waves. Using $R = 287 \text{ m}^2 \text{ s}^{-2} \text{ deg}^{-1}$ and $T_4 = 273^\circ \text{K}$ we find from (11.5.23) that $c = \pm 280 \text{ m s}^{-1}$. These waves therefore move almost as fast as the sound waves in the atmosphere.

The second special case is one which may be obtained by replacing the lower boundary condition (11.5.17) by the much simpler condition $\omega_4 = 0$, $p = p_4$. This condition can be justified as an approximation, because

\[
\omega = \frac{dp}{dt} = \frac{\partial p}{\partial t} + \nabla \cdot vp + w \frac{\partial p}{\partial z} \tag{11.5.24}
\]

Since the observed wind is approximately geostrophic we have $\nabla \cdot vp \approx 0$. It is furthermore possible to show that $|\partial p/\partial t| \ll gpw$ by using observed tendencies. We have therefore approximately
\[
\omega = g^2 w
\]  
(11.5.25)

At the lowest boundary where \( w = 0 \) we have thus \( \omega \approx 0 \). If we adopt this boundary condition, we may find the effects of it by setting \( \tilde{p}_4 = 0 \) in (11.5.21) because the condition (11.5.17) reduces to \( \omega' = 0 \) if \( \tilde{p}_4 = 0 \). We find then from the second equation in (11.5.21) that

\[
a_1 + a_3 = 0
\]  
(11.5.26)

which, among other things, means that the total divergence in a whole atmospheric column is equal to zero.

From the first equation in (11.5.21) we find

\[
(2c^2 - \sigma_2^2 p^2) a_1 = 0
\]  
(11.5.27)

which leads to

\[
c = \pm \sqrt{\frac{\sigma_2^2 p^2}{2}}
\]  
(11.5.28)

The waves, which we obtain as solutions in this special case, are determined entirely by the internal stratification (\( \sigma_2 \)), and they are called internal gravity waves. The speed of these waves is determined by (11.5.28). \( \sigma_2 \) may be calculated from observations, and it is found that the value of \( \sigma_2 \) is of the following order of magnitude:

\[
\sigma_2 \approx 2 \times 10^{-6} \text{ mks-units}
\]  
(11.5.29)

and that \(|c| \approx 50 \text{ m s}^{-1}\). The internal gravity waves move therefore with a considerably smaller speed than the external gravity waves.

We emphasize that the speeds given for the external and internal gravity waves should be considered as order of magnitude estimates. This is especially true for the internal gravity waves which depend on the numerical values of \( \sigma_2 \).

We stress furthermore that the external gravity waves are characterized by a non-vanishing divergence in a vertical column because \( \omega_4 \neq 0 \). They can be eliminated by adopting the simplified lower boundary condition \( \omega_4 = 0, p = p_4 \).
In the general case we must solve the equation obtained by setting the determinant to (11.5.21) equal to zero. Let us introduce the notations (see (11.5.28) and (11.5.23))

$$c^{2}_{IN} = \frac{\sigma^{2} p^{2}}{2} \quad \text{and} \quad c^{2}_{EX} = \frac{2p}{\rho_{4}} \quad \text{(11.5.30)}$$

The determinant may then be written in the form

$$c^{4} - (c^{2}_{EX} + 3c^{2}_{IN}) c^{2} + c^{2}_{IN} c^{2}_{EX} = 0 \quad \text{(11.5.31)}$$

and we find

$$c^{2} = \frac{1}{2} \left[ \left( c^{2}_{EX} + 3c^{2}_{IN} \right) \pm \sqrt{c^{4}_{EX} + 9c^{4}_{IN} + 2c^{2}_{IN} c^{2}_{EX}} \right] \quad \text{(11.5.32)}$$

It is easily seen that both values in (11.5.32) are positive, and that we will get four values of \(c\) from (11.5.32). We note especially that there is no possibility for instability. Two of the four values from (11.5.32), namely those corresponding to the plus sign, will have speeds corresponding to the external waves. Using the same numerical values as before we find in this case

$$c_{1,2} = \pm 290 \text{ m s}^{-1}$$

The minus sign in (11.5.32) will on the other hand give values of \(c\) which are in qualitative agreement with the speed of the internal gravity waves. We find

$$c_{3,4} = \pm 59 \text{ m s}^{-1}$$

11.6 Inertia Waves

The physical factor which more than anything else influences the atmospheric flow and makes it distinctly different from the types considered in sections 11.4 and 11.5 is the rotation of the earth. We shall again start by considering a very special case in which we find one aspect of the earth's rotation in a pure form.

Let us consider the possible wave motion which may exist if we assume horizontal motion in a case where the pressure force can be neglected. Assuming again that the basic state is a state of no motion we find from the equations of motion

$$\frac{\partial u'}{\partial t} = f v' \quad \text{(11.6.1)}$$
\[ \frac{d\gamma'}{dt} = -fu' \] (11.6.2)

which form a closed system.

It is straightforward to obtain solutions to (11.6.1) - (11.6.2) if we assume disturbances of the types (11.5.19). Assuming \( f = f_0 \), we get

\[ ikc \hat{u} + f_0 \hat{v} = 0 \] (11.6.3)

\[ f_0 \hat{u} - ikc \hat{v} = 0 \]

from which we find, setting the determinant equal to zero

\[ c = \pm \frac{f_0}{k} \] (11.6.4)

The values given by (11.6.4) are called the inertia wave speeds. The motion of these waves are, in contradistinction to the sound waves and the external and internal gravity waves, dependent upon the wave number \( k \). We find from (11.6.4) using \( k = 2\pi/L \) that

\[ c = \pm \frac{f_0 L}{2\pi} \] (11.6.5)

Assuming \( L = 1000 \text{ km} = 10^6 \text{ m} \) we find for \( f_0 = 10^{-4} \text{ s}^{-1} \) that \( c = \pm 15.9 \text{ m s}^{-1} \).

These waves are therefore slowly moving waves for reasonably small values of the wavelength. It is true that \( c \), computed from (11.6.5), will become very large for large values of \( L \), but the assumption of a constant value of \( f \) may not hold when \( L \) is large.

11.7 Inertia-gravity Waves

It goes without saying that the gravity waves and the inertia waves have been considered in pure form in the preceding sections, but that they will normally exist together just as internal and external gravity waves do (see section 11.5).

We shall in this section consider a model which permits internal gravity waves and inertia waves at the same time. This can be demonstrated using the two-level model with a basic state of rest just as in section 11.5, but adding now the
Coriolis force to the system. Restricting the variation of the perturbation dependent variables to the $x$- and $t$-variables we find the following perturbation equations:

\[
\frac{du'_1}{dt} = -\frac{d\phi'_1}{dx} + f_0 v'_1 \\
\frac{dv'_1}{dt} = -f_0 u'_1 \\
\frac{du'_3}{dt} = -\frac{d\phi'_3}{dx} + f_0 v'_3 \\
\frac{dv'_3}{dt} = -f_0 u'_3
\]

(11.7.1)

\[
\omega'_2 = -P \frac{du'_1}{dx} \\
-\omega'_2 = -P \frac{du'_3}{dx} \\
-\frac{1}{P} \frac{d(\phi'_1 - \phi'_3)}{dt} + \frac{c_2}{2} \omega'_2 = 0.
\]

We have already incorporated the boundary condition $\omega_4 = 0$, excluding the external gravity waves, in the system of equations (11.7.1). The type of disturbances will be of the usual form

\[
(\cdot') = (-\cdot) e^{ik(x-ct)}
\]

(11.7.2)

Inserting (11.7.2) in the first two equations in (11.7.1) we find

\[
-ikc a_1 - f_0 \varphi'_1 + ik \hat{\varphi}_1 = 0
\]

(11.7.3)

\[
-ikc \varphi'_1 + f_0 a_1 = 0
\]

(11.7.4)

Substituting from (11.7.4) in (11.7.3) and eliminating $\varphi'_1$, we find

\[
\hat{\varphi}_1 = \left( c - \frac{c_1^2}{c} \right) a_1
\]

(11.7.5)

where $c_1^2 = f_0^2 / k^2$, the square of the pure inertia wave speed.
Since the third and fourth equations in (11.7.1) are completely analogous to the first and second equations we find
\[ \hat{\Phi}_3 = \left(c - \frac{c^2}{c}\right) \Phi_3 \] (11.7.6)

From the fifth and sixth equations in (11.7.1) we find that
\[ \hat{\omega}_2 = -P \, ik \, a_1 \] (11.7.7)
\[ \hat{\omega}_2 = -P \, ik \, a_3 \]

Addition and subtraction of these equations give
\[ a_1 + a_3 = 0 \] (11.7.8)

and
\[ \hat{\omega}_2 = -P \, ik \frac{1}{2} (a_1 - a_3) = -P \, ik \, a_1 \] (11.7.9)

where the last expression in (11.7.9) is obtained by use of (11.7.8). It follows furthermore by substitution of (11.7.8) into (11.7.5) and (11.7.6) that
\[ \hat{\Phi}_1 + \hat{\Phi}_3 = 0 \] (11.7.10)

The results from (11.7.5), (11.7.9) and (11.7.10) are now substituted into the last equation in the system (11.7.1), and we get
\[ \frac{iKe}{P} \, 2\hat{\Phi}_1 + \bar{\sigma}_2 \hat{\omega}_2 = 0 \] (11.7.11)

or
\[ \left[ \frac{iKe}{P} \, 2 \left(c - \frac{c^2}{c}\right) \bar{\sigma}_2 \, P \, ik \right] a_1 = 0 \] (11.7.12)

which will be satisfied provided
\[ c^2 - \sigma^2 - \sigma^2_{IN} = 0 \] (11.7.13)

where we have adopted the notation used in (11.5.30) for the internal gravity waves. We get therefore
\[ c = \pm \sqrt{\frac{c_I^2 + c_{IN}^2}{c_I^2}} \] (11.7.14)

which is the speed of the inertia-gravity waves. We notice that \( c \) will be almost equal to \( c_{IN} \) when the wavelength is small, but that the term \( c_I^2 \) gains increasing importance as the wavelength increases.

We mention finally that it is possible to include external and internal gravity waves and inertia waves in the same system and obtain a solution using the two-level model, but the details of the calculation will not be reproduced here.

11.8 **Rossby Waves (Barotropic Waves)**

The various types of waves treated so far in this chapter have either been uninfluenced by the rotation of the earth, or it has been assumed that the Coriolis parameter was constant. It turns out in fact that the rotation of the earth is very important in determining the types of flow which are dominant in the atmosphere, and, more importantly, the south-north variation of the Coriolis parameter is of particular significance. The facts stated above were first brought forward by C. G. Rossby in 1939. In order to demonstrate the effect of the meridional variation of \( f \), the Coriolis parameter, we shall take the particularly simple example used by Rossby.

We start by considering the vorticity equation. As has been pointed out before, we find from observational studies that the general order of magnitude of the divergence in the atmosphere is smaller than the order of magnitude of the vorticity, i.e. \( |\mathbf{\nabla} \cdot \mathbf{v}_h| \ll |\zeta| \). This fact is closely related to the fact that the vertical motion is one or two orders of magnitude smaller than the horizontal motion, i.e. \( |w| \ll |\mathbf{\nabla} \cdot \mathbf{v}_h| \). The essence of the studies mentioned above is that the major component of the horizontal flow can be considered as non-divergent, and it can be described by a stream-function.

The motion of the Rossby waves can now be demonstrated in a pure form if we disregard the horizontal divergence and the vertical velocity altogether. We assume, in other words, that the flow is horizontal and non-divergent. The vorticity equation is then

\[ \frac{d(\zeta + f)}{dt} = 0 \] (11.8.1)

or
\[ \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + \beta v = 0 \]  

(11.8.2)

where

\[ \beta = \frac{\partial f}{\partial y} = \frac{\partial (2a \cos \varphi)}{\partial \varphi} = \frac{2a \cos \varphi}{a} \]  

(11.8.3)

\( \beta \) is called the Rossby parameter.

We consider now a co-ordinate system which is a local Cartesian system except that the real variation of \( f \) is replaced by

\[ f = f_o + \beta y \]  

(11.8.4)

where \( f_o \) is the value of \( f \), \( y = 0 \), and \( \beta \) by assumption is constant. A horizontal plane in which (11.8.4) is valid is called a beta-plane.

Let us now assume that the basic state in the beta-plane is a constant velocity \( U \) in the \( x \)-direction, i.e. \( \bar{u} = U, \bar{v} = 0 \). We note that this is a permissible basic state because \( \zeta = 0 \), and all terms in (11.8.2) are therefore identically zero.

Let us next consider a perturbation, where for simplicity we shall assume that it depends on \( x \) and \( t \) only. Under these circumstances we have

\[ \zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \frac{\partial v'}{\partial x} \]  

(11.8.5)

and the perturbation equation obtained from (11.8.2) becomes

\[ \frac{\partial}{\partial t} \left( \frac{\partial v'}{\partial x} \right) + U \frac{\partial}{\partial x} \left( \frac{\partial v'}{\partial x} \right) + \beta v' = 0 \]  

(11.8.6)

Assuming the usual form

\[ v' = v e^{ik(x-ct)} \]  

(11.8.7)

we find from (11.8.6) that

\[ -ik c \cdot ik \phi + (ik) \cdot (ik) U \phi + v = 0 \]  

(11.8.8)

or

\[ -ik c \cdot ik \phi + (ik)^2 U \phi + v = 0 \]
\[ c = U - \frac{\beta}{k^2} \quad (11.8.9) \]

The speed of the waves is given by (11.8.9), which is called the Rossby wave formula. Contrary to all the waves considered previously we find that the Rossby waves move in a definite direction, while the other waves always had a double sign. It is furthermore seen that the wave will move westward relative to the basic current \( U \).

Let us assume that \( U > 0 \). We find then that the wave will move from west to east \( (c > 0) \) as long as

\[ U - \frac{\beta}{k^2} > 0 \quad (11.8.10) \]

while the motion will be in the opposite direction \( (c < 0) \) when

\[ U - \frac{\beta}{k^2} < 0 \quad (11.8.11) \]

The limiting case of the stationary Rossby wave is found when

\[ \frac{\beta}{k_S^2} = U \quad (11.8.12) \]

where \( k_S \) is the so-called stationary wave number. It is customary to use a stationary wavelength \( L_S = 2\pi/k_S \) determined from the relation

\[ L_S = 2\pi \sqrt{\frac{U}{\beta}} \quad (11.8.13) \]

We may easily estimate a couple of values of \( L_S \). Let us first find a typical value of \( \beta \) in middle latitudes, say 45\(^\circ\)N. We find from (11.8.3) that

\[ \beta \approx 16 \times 10^{-12} \text{ m}^{-1} \text{ s}^{-1} \]

For \( U = 20 \text{ m s}^{-1} \), typical of winter conditions at 500 mb, we get

\[ L_S \approx 7.025 \times 10^6 \text{ m} = 7025 \text{ km} \]

while a value of \( U = 10 \text{ m s}^{-1} \), more typical of summer conditions, gives \( L_S \approx 4.967 \times 10^6 \text{ m} = 4967 \text{ km} \). Rossby waves with a wavelength shorter than these values move eastward while waves with a larger wavelength will move westward (retrogressive waves).
It was pointed out by Rossby that the motion of the waves in the real atmosphere at about 500 mb is in agreement with (11.8.9), at least qualitatively, provided the wavelength is not too large. It seems therefore that the real atmosphere considered around the middle of the troposphere behaves as if it were in horizontal, non-divergent motion. The explanation of this fact will be considered later, but it is worthwhile to stress that the model, the dynamics of which are given in (11.8.1), in spite of the extreme assumptions, describes nevertheless the mid-tropospheric flow to the first approximation.

The quantity $\beta/k^2$ is sometimes denoted

$$c_R = \frac{\beta}{k^2}$$  \hspace{1cm} (11.8.14)

where $c_R$ is called the Rossby speed. At 45°N we find, approximately

$$c_R = \frac{16 \times 10^{-12}}{4\pi^2} \times L^2 \approx 0.4 \, l^2$$  \hspace{1cm} (11.8.15)

where $l$ is the wavelength measured in units of $10^6 \, m = 1000 \, km$.

11.9  
**Baroclinic Waves (Quasi-geostrophic Model)**

The vorticity equation used in section 11.8 is extremely simplified. According to our discussion of the complete vorticity equation (section 6.11) in pressure coordinates, we found from observational studies that the next term to include in the vorticity equation considering the general order of magnitude is $f \nabla \cdot \mathbf{v}$. It can therefore be assumed that the equation

$$\frac{d\zeta}{dt} + \nabla \cdot (\zeta + f) = -f \nabla \cdot \mathbf{v} = f \frac{d\omega}{dp}$$  \hspace{1cm} (11.9.1)

will be a closer approximation to the atmospheric flow than the barotropic vorticity equation (11.8.1). Equation (11.9.1) is widely used in atmospheric dynamics and numerical weather prediction, but it should be mentioned immediately that (11.9.1) must be combined with at least one other equation in order to form a closed system, since it contains the additional variable $\omega$.

We refer now to our discussion of the geostrophic wind and the geostrophic vorticity (section 8.3). It was found at that time that the geostrophic approximation is acceptable as a first approximation for the calculation of the horizontal wind and
the vertical components of vorticity, but not for the calculation of the horizontal divergence. If we adopt this approximation we may write (11.9.1) in the form

\[ \frac{\partial \xi'}{\partial t} + \vec{v}' \cdot \nabla (\xi' + f) = f \frac{\partial \omega'}{\partial p} \quad (11.9.2) \]

where \( \xi' \approx f^{-1} \nu' \phi \) and \( \vec{v}' = f^{-1} \nabla \times \nabla \phi \). (11.9.2) has then two dependent variables: \( \phi \) and \( \omega' \). Another equation with the same two dependent variables can be found in the thermodynamic energy equation under the adiabatic assumption. We have (see (11.5.7))

\[ \frac{d}{dt} \left( \frac{\partial \phi'}{\partial p} \right) + \vec{v}' \cdot \nabla \left( \frac{\partial \phi'}{\partial p} \right) + \sigma \omega = 0 \quad (11.9.3) \]

where we have also introduced the geostrophic wind in the horizontal advection term. (11.9.2) and (11.9.3) form a closed system, which forms the basic equations for the so-called quasi-geostrophic model, widely used in numerical prediction in various forms.

In this section we shall only be interested in a perturbation analysis of (11.9.2) and (11.9.3). We consider then a basic state in which \( \vec{u} = \vec{U}(p) \), \( \vec{v} = \vec{\omega} = 0 \). The basic state is therefore a zonal current with vertical wind shear. It will be assumed that \( \vec{U}(p) \) is geostrophic. We have therefore:

\[ fU = -\frac{\partial \phi}{\partial y} \quad (11.9.4) \]

and, in particular,

\[ f \frac{dU}{dp} = -\frac{d}{dy} \left( \frac{\partial \phi}{\partial p} \right) \quad (11.9.5) \]

where \( \phi = \phi(y,p) \) is the geopotential in the basic state.

We note that the basic state just defined is permissible, since all terms in (11.9.2) and (11.9.3) become identically equal to zero. We have therefore the following perturbation equations:

\[ \frac{\partial \xi'}{\partial t} + \vec{U} \cdot \frac{\partial \xi'}{\partial x} + \beta v' g = f \frac{\partial \omega'}{\partial p} \quad (11.9.6) \]

\[ \frac{d}{dt} \left( \frac{\partial \phi'}{\partial p} \right) + \vec{U} \frac{d}{dx} \left( \frac{\partial \phi'}{\partial p} \right) + \vec{v}' \cdot \nabla \left( \frac{\partial \phi'}{\partial p} \right) + \vec{g} \cdot \nabla \left( \frac{\partial \phi'}{\partial p} \right) + \sigma \omega' = 0. \quad (11.9.7) \]
The perturbation problem given by (11.9.6) and (11.9.7) has been solved when $U = U(p)$ is given as a linear function of $p$ and $\bar{\sigma} = \bar{\sigma}(p) = \text{const.}$ These investigations go beyond the scope of this compendium, but we shall solve the system (11.9.6) and (11.9.7) using the two-level model introduced in section 11.5. Referring to Figure 11.2 we apply (11.9.6) at levels 1 and 3 and (11.9.7) at level 2. In order to avoid too many subscripts and we shall drop the "prime" and the subscript $g$ in the equations. We get

\[
\frac{d\xi_1}{dt} + U_1 \frac{d\xi_1}{dx} + \beta v_1 = \frac{f}{P} \omega_2
\]  
(11.9.8)

\[
\frac{d\xi_2}{dt} + U_3 \frac{d\xi_3}{dx} + \beta v_3 = \frac{f}{P} \omega_2
\]  
(11.9.9)

where we have incorporated the boundary conditions $\omega = 0$ at $p = 0$ and $p = p_4$ thereby eliminating the external gravity waves.

We shall now introduce the notations:

\[
b_* = \frac{1}{2} (b_1 + b_3)
\]  
(11.9.10)

\[
b_T = \frac{1}{2} (b_1 - b_3)
\]

where subscripts 1 and 3 refer to levels 1 and 3.

Adding and subtracting (11.9.8) and (11.9.9) we find (after division by 2) and using (11.9.10)

\[
\frac{d\xi_*}{dt} + U_* \frac{d\xi_*}{dx} + U_T \frac{d\xi_T}{dx} + \beta v_* = 0
\]  
(11.9.11)

\[
\frac{d\xi_T}{dt} + U_* \frac{d\xi_T}{dx} + U_* \frac{d\xi_*}{dx} + \beta v_T = \frac{f}{P} \omega_2
\]  
(11.9.12)

We apply next (11.9.7) at level 2 assuming that $U_2 \approx U_*$ and $v_2 \approx v_*$ and get

\[- \frac{2}{F} \frac{d\phi_T}{dt} - \frac{2}{F} U_* \frac{d\phi_T}{dx} + \frac{2}{P} U_T f \beta v_* + \bar{\sigma}_2 \omega_2 = 0\]  
(11.9.13)

where we have made use of (11.9.5) and have approximated $(dU/dp)_2$ by finite differences. As before we shall consider perturbations depending on $x$ and $t$, disregarding the meridional direction. We have thus
\[ \zeta_* = \frac{1}{f} \frac{\partial^2 \phi_*}{\partial x^2}; \quad \zeta_T = \frac{\partial^2 \phi_T}{\partial x^2} \]  
(11.9.14)

We note furthermore that

\[ v_* = \frac{1}{f} \frac{\partial \phi_*}{\partial x}; \quad v_T = \frac{1}{f} \frac{\partial \phi_T}{\partial x} \]  
(11.9.15)

When these expressions are introduced in (11.9.11) - (11.9.13) and \( \omega_2 \) is eliminated we get

\[ \frac{\partial}{\partial t} \left( \frac{\partial^2 \phi_*}{\partial x^2} \right) + U_* \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_*}{\partial x^2} \right) + U_T \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_T}{\partial x^2} \right) + \beta \frac{\partial \phi_*}{\partial x} = 0 \]  
(11.9.16)

\[ \frac{\partial}{\partial t} \left( \frac{\partial^2 \phi_T}{\partial x^2} \right) + U_* \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_T}{\partial x^2} \right) + U_T \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_*}{\partial x^2} \right) + \beta \frac{\partial \phi_T}{\partial x} = 0 \]

\[ q^2 \left[ \frac{\partial \phi_T}{\partial t} + U_* \frac{\partial \phi_T}{\partial x} - U_T \frac{\partial \phi_*}{\partial x} \right]; \quad q^2 = \frac{2r^2}{\sigma p^2} \]  
(11.9.17)

The perturbation will have the form

\[ ( - ) e^{ik(x-ct)} \]  
(11.9.18)

The expressions (11.9.18) for \( \phi_* \) and \( \phi_T \) are introduced in (11.9.16) and (11.9.17). The resulting equations are two linear homogeneous equations in \( \hat{\phi}_* \) and \( \hat{\phi}_T \). They are:

\[ (c - U_* + c_R) \hat{\phi}_* - U_T \hat{\phi}_T = 0 \]  
(11.9.19)

\[ \left( \frac{4}{k^2} - 1 \right) U_T \hat{\phi}_T + \left( c - U_* + c_R + \frac{q^2}{k^2} (c - U_*^2) \right) \hat{\phi}_T = 0 \]  
(11.9.20)

where we have used (as before) the notation \( c_R = \beta/k^2 \). We note that the wave speed \( c \) appears in the combination \( (c - U_*) \) everywhere. We shall consequently introduce the notation

\[ x = c - U_* \]  
(11.9.21)

where \( x \) then is the speed of the wave relative to the vertically averaged wind. The determinant of (11.9.19) and (11.9.20) must vanish if we are to have non-trivial solutions. This condition will give us an equation for the determination of \( x \) in
terms of the other parameters in the problem. We get:

\[ (1 + \frac{q^2}{k^2})x^2 + (2 + \frac{q^2}{k^2})c_R x + \left[ c_R^2 - \left(1 - \frac{q^2}{k^2}\right)u_T^2 \right] = 0 \quad (11.9.22) \]

The solutions to (11.9.22) are

\[ x = -\frac{(2 + \frac{q^2}{k^2})c_R}{2\left(1 + \frac{q^2}{k^2}\right)} \pm \sqrt{\frac{(\frac{q^4}{k^4}) c_R^2 + 4 \left(1 - \frac{q^4}{k^4}\right) u_T^2}{2\left(1 + \frac{q^2}{k^2}\right)}} \quad (11.9.23) \]

which gives the wave speed of the waves which are possible in the quasi-geostrophic two-level model.

Our first concern will be the question of stability. As mentioned in section 11.2 we will have instability if \( c \) has a positive imaginary part, \( c_+ > 0 \), but stability if \( c_- < 0 \). It is seen from (11.9.22) that the possible complex wave speeds will form conjugate complex numbers. If we have a complex value of \( x = c - u_T \) we will therefore always have an instability. It is seen from (11.9.23) that the condition for instability is that

\[ \frac{q^4}{k^4} c_R^2 + 4 \left(1 - \frac{q^4}{k^4}\right) u_T^2 < 0 \quad (11.9.24) \]

This condition can not be satisfied if

\[ q < k \quad (11.9.25) \]

or

\[ L < \frac{2\pi}{q} = L_c \quad (11.9.26) \]

It is thus seen that sufficiently short waves are always stable. We have

\[ L_c^2 = \frac{4\pi^2 \bar{\sigma}^2 \bar{p}^2}{2f^2} \quad (11.9.27) \]

Using \( \bar{\sigma}^2 = 2 \times 10^{-6} \) mke-units, \( \bar{p} = 500 \) mb, \( f = 10^{-4} \) s\(^{-1} \) we find \( L_c = \pi \times 10^6 \) m = 3142 km, which is typical for middle latitudes. It is interesting to note that \( L_c \) will be larger in low latitudes because of the variation of \( f \).

We shall in the following assume that \( k < q \) or \( L > L_c \). It is in this region that there exists a possibility for unstable solutions. We find then from (11.9.24)
that the condition for instability is

\[ u_T^2 > \frac{\frac{a^4}{k^4} c_R^2}{4 \left( \frac{a^4}{k^4} - 1 \right)} \]  \hspace{1cm} (11.9.28)

or

\[ u_T > \frac{1}{2} \frac{\frac{a^2}{k^2} c_R}{\sqrt{\frac{a^4}{k^4} - 1}} = u_{T,c} \]  \hspace{1cm} (11.9.29)

assuming that \( u_T > 0 \) as is normally the case in the atmosphere.

---

![Figure 11.3](image-url)
It is thus seen that if \( U_T > U_{T,C} \) as defined by (11.9.29) we will have instability. The critical thermal wind \( U_{T,C} \) is plotted in Figure 11.3 as a function of the wavelength \( \lambda = 2\pi/k \) measured in the unit \( 10^6 \) m = 1000 km. In the construction of Figure 11.3 we have used the same numerical values as in finding \( L_C \) and \( \beta = 16 \times 10^{-12} \) m\(^{-1}\) s\(^{-1}\). Figure 11.3 shows the stable region for \( L < L_C \) for all values of \( U_T \). When \( L > L_C \) we find stability according to Figure 11.3 when \( U_T < 4 \) m s\(^{-1}\). However, if \( U_T > 4 \) m s\(^{-1}\) there will exist a band of wavelengths for which instability will set in. The main result of this analysis is therefore that an intermediate band of wavelength will be unstable for sufficiently large values of the wind shear.

The next question which must be investigated is naturally where typical atmospheric values of \( U_T \) are located on the diagram. The vertical wind shear in the atmosphere is normally given as \( dU/dz \). We find:

\[
\frac{dU}{dz} = \frac{dp}{dz} \frac{dU}{dp} = -\frac{g}{RT} \left( -\frac{2}{P} U_T \right) = \frac{2g}{RT} U_T \tag{11.9.30}
\]

or

\[
U_T \approx 3.66(dU/dz) \tag{11.9.31}
\]

where \( dU/dz \) is measured in m s\(^{-1}\) km\(^{-1}\) in the last equation.

Typical values of \( dU/dz \) are around 2 m s\(^{-1}\) km\(^{-1}\) giving \( U_T = 7.3 \) m s\(^{-1}\). This typical value is given by the dashed line in Figure 11.3, but it should be realized that there are large variations in \( dU/dz \) with the seasons giving large values in winter and small values in summer. We may conclude that the westerlies in middle latitudes are always unstable with respect to disturbances of the type considered here. The results obtained from models considerably more complicated than the two-parameter model analysed here are in agreement with the statement made above.

It is of considerable interest to investigate which physical factors determine the shape of the critical curve given in Figure 11.3. We note from (11.9.26) that the asymptote determined by \( L_C \) is proportional to \( \sqrt{\sigma_2} \). It follows that the short wave cut-off, i.e. the value of \( L \) below which the waves become stable, will be at a very small wavelength when \( \sigma_2 \) becomes small, i.e. when the actual lapse rate approaches the dry-adiabatic lapse rate. If \( \sigma_2 = 0 \), i.e. \( \gamma = \gamma_d \), we find \( L_C = 0 \). The static stability is therefore responsible for the baroclinic stability at sufficiently short wavelengths.
There is also a long wave cut-off in Figure 11.3. It is due to the beta-effect. We may see this by setting \( c_R = 0 \) in (11.9.22) and obtain

\[
(1 + \frac{q^2}{k^2}) x^2 - (1 - \frac{q^2}{k^2}) u_T^2 = 0
\]

(11.9.32)

The solution is

\[
x = \pm u_T \sqrt{\frac{k^2 - q^2}{k^2 + q^2}}
\]

(11.9.33)

and it is seen that there will be instability, regardless of the value of \( U_T \), as long as \( k < q \), i.e. \( L > L_c \), as defined by (11.9.27). We may conclude that the long wave cut-off is due to the beta-effect which acts as a stabilizer on the long waves.

The next question which will be investigated is the degree of instability which exists in the unstable region. We shall introduce the so-called e-folding time as a measure of the degree of instability. Considering (11.2.7) we find that the amplitude of the unstable wave is

\[
B e^{k_0 t}
\]

(11.9.34)

At \( t = 0 \) the amplitude is \( B \). The e-folding time is the time it takes for the amplitude to increase from \( B \) to \( B e \). We find that the e-folding time is

\[
t_e = \frac{1}{kc_1} = \frac{L}{2\pi c_1}
\]

(11.9.35)

We note that \( t_e \to \infty \) as \( c_1 \to 0 \). Small values of \( t_e \) correspond to large values of \( c_1 \).

We find from (11.9.23) that

\[
t_e = \frac{1}{k} \sqrt{\frac{2(1 + \frac{q^4}{k^4})}{4U_T^2(k_4^4 - 1) - \frac{q^4}{k^4} c_R^2}}
\]

(11.9.36)

Figure 11.4 shows \( t_e \) (in units of days) as a function of wavelength using the same numerical values as before and \( U_T = 10 \right m s^{-1} \). We note that the minimum in \( t_e \) (i.e. the maximum instability) occurs around 4500 km, and that the amplitude will increase by a factor \( e \) in about 1.5 days. An e-folding time of about 1 day is quite
possible for values of $U_T$ typical of winter. The order of magnitude found for the e-folding time is in agreement with the experience gained by synoptic meteorologists that a rather insignificant wave may develop in a day or two.

We shall next investigate the wave speed in (11.9.23). The behaviour is most interesting for values of $U_T$ for which a band of waves is unstable. It is seen that we obtain two values of $x$, and therefore $c = x + U_*$, in the stable regions, but only one value in the unstable region. The two values in the stable region become identical as we approach the neutral (or critical) curve. These conditions are illustrated in Figure 11.5 where we have plotted $c_1$ (corresponding to the plus sign in (11.9.23)) and $c_2$ as a function of wavelength using $U_* = 10$ m s$^{-1}$ while the other variables are identical to those entering Figures 11.3 and 11.4. It is of special interest to note that $c_1$ for short waves and $c_2$ for long waves are of the same type as the non-divergent Rossby wave as treated in section 11.8, i.e. $c = U_* - \beta/k^2$. This is also seen in Figure 11.5 where we have plotted the Rossby wave speed as the dashed curve with circles.
Figure 11.5
In addition to the Rossby type wave we will also have the other solution whose speed is $c_1$ for large values of $L$. It is seen that this wave moves with a small positive speed less than $U_*$ for large values of $L$. It is of interest to note that the asymptotic value of $x_1$ is $-\beta/q^2$ when $k \to 0$ ($L \to \infty$). One can see this from (11.9.23). If we let $k \to 0$ in the expression for $x_1$ we obtain an expression of the type $0/0$. However, a double use of l'Hospital's rule gives the expression above. We have therefore

$$\lim_{k \to 0} c_1 = U_* - \frac{\beta}{q^2} = 6 \text{ m s}^{-1}$$

In addition to the remarks made above we may also obtain an understanding of the two types of solution by reconsidering the perturbation analysis of (11.9.11), (11.9.12), and (11.9.13) using the very simple basic state of $U_* = U_T = 0$. The three equations become

$$\frac{\partial x_*}{\partial t} + \beta v_* = 0 \quad (11.9.37)$$

$$\frac{\partial x_T}{\partial t} + \beta v_T = \frac{f_0}{p} \omega_2 \quad (11.9.38)$$

$$-\frac{2}{p} \frac{\partial \phi_T}{\partial t} + \omega_2 = 0 \quad (11.9.39)$$

It is seen that the three equations in this simple case become uncoupled in such a way that the first equation contains all information about the mean flow, while the remaining two equations pertain to the thermal flow. Using perturbations of the same type as before (see (11.9.18)) we find from (11.9.37) that $c = -\beta/k^2$, while (11.9.38) and (11.9.39) give the solution $c = -\beta/(k^2 + q^2)$. It is thus obvious that one of the solutions is mainly a pure Rossby wave and is to be found in the mean flow ($c = c_2$ for large $L$) while the other solution is related to the thermal flow ($c = c_1$ for large $L$). The limiting value for $k \to 0$ ($L \to \infty$) of the thermal solution is, as in the general case, $-\beta/q^2$.

The relations discussed above may also be illustrated in the following way. We have found the phase speed $c$ by using the condition under which the equations (11.9.19) and (11.9.20) have non-trivial solutions. When $c$ has been obtained (see (11.9.23)), we may find the ratio between the unknowns $\phi_*$ and $\phi_T$ from either (11.9.19) or (11.9.20). We get

$$\frac{\ddot{\phi}_*}{\ddot{\phi}_T} = \frac{U_T}{c - U_* + c_R} \quad (11.9.40)$$
The discussion of (11.9.40) is most easily done if we distinguish between the stable and unstable cases. Considering first the stable case, we have real quantities everywhere in (11.9.40), and it is a straightforward matter to calculate the ratio \( \hat{\phi}_*/\hat{\phi}_T \) in the two cases \( c = c_1 \) and \( c = c_2 \). The result is shown in Figure 11.6. For the short stable waves we find that \( c_1 \) gives a ratio which is positive, which means that \( \hat{\phi}_* \) and \( \hat{\phi}_T \) are in phase, while \( c_2 \) results in a negative ratio indicating that \( \hat{\phi}_* \) and \( \hat{\phi}_T \) are one half wavelength out of phase. The result for the long stable waves indicates that the amplitude of the wave in the mean field is much larger than the amplitude in the thermal field (see curve (2) corresponding to \( c_2 \), for the Rossby type wave) while the opposite is the case for the other wave with \( c = c_1 \) (see curve (1)) which may therefore be called the thermal wave.

We shall next turn our attention to the unstable case, where \( c = c_T + ic_1 \). The values of \( \hat{\phi}_* \) and \( \hat{\phi}_T \) are then also complex. Let \( \hat{\phi}_* = \hat{\phi}_*(R) + i \hat{\phi}_*(I) \) and \( \hat{\phi}_T = \hat{\phi}_T(R) + i \hat{\phi}_T(I) \). When we substitute these expressions in (11.9.40), equate the real and the imaginary parts and solve for \( \phi_T(R) \) and \( \phi_T(I) \), we get

\[
\phi_T(R) = \frac{c_T - U_* + c_R}{U_T} \phi_*(R) - \frac{c_1}{U_T} \phi_*(I)
\]
\[
\phi_T(I) = \frac{c_1}{U_T} \phi_*(R) + \frac{c_T - U_* + c_R}{U_T} \phi_*(I)
\]

(11.9.41)

Since we can only obtain the relative solution given in (11.9.41) we can without loss of generality set \( \phi_*(R) = A_* \) and \( \phi_*(I) = 0 \). This means that we count the abscissa from the ridge in the \( \phi_* \)-wave (see (11.2.4) and (11.2.6)). We find then that

\[
A_T = \sqrt{\phi_T(R)^2 + \phi_T(I)^2} = A_* \sqrt{(c_T - U_* + c_R)^2 + c_1^2 / U_T}
\]

(11.9.42)

and

\[
\tan \delta_T = \frac{\phi_T(I)}{\phi_T(R)} = \frac{c_1}{c_T - U_* + c_R}
\]

(11.9.43)

From (11.9.23) we obtain

\[
c_T = U_* + c_R = \frac{\gamma^2}{2(1 + \frac{\gamma^2}{k^2})} c_R > 0
\]

(11.9.44)
Figure 11.6
and it is then obvious that $\tan \delta_T > 0$ for the unstable wave where $c_i > 0$, while $\tan \delta_T < 0$ for $c_i < 0$ (damping waves). As an example we may take $L = 5000$ km in which case we have $c_T = 3.57$ m s$^{-1}$ and $c_i = 5.48$ m s$^{-1}$. We get in that case that $A_/A_*= 0.65$ while $\delta_T = 57^0$ or 0.16 of a wavelength (see Figure 11.7).

The main result is that the thermal wave $\phi_T$ is lagging behind the mean wave $\phi_*$ in such a way that we have cold air advection into the troughs and warm air advection into the ridges. This feature of the amplifying wave is also in good agreement with synoptic experience, and we may thus conclude that the relatively simple two-level, quasi-geostrophic model is able to describe many of the observed features of the large scale flow in the troposphere.

Figure 11.7
Chapter XII

NUMERICAL PREDICTION

12.1 Introduction

The prediction problem has been considered in principle in Chapter III and again in Chapter VII. In each of the two chapters we described the main parts of the computational cycle needed to advance the prediction one time step in such a way that, by repeated use of the cycle, we can in principle make predictions over a time period as long as we desire.

A system of equations very similar to those developed in Chapter III was used by L.F. Richardson (Weather Prediction by Numerical Process, 1922) to make the first experiment in numerical weather prediction. It became evident from this experiment that the computational work needed to make a numerical prediction for, say, 24 hours was so large that computers much larger and much faster than those available to Richardson were needed for it to be possible to make the prediction in a sufficiently short time for the forecast to be useful. In addition, it was obvious that observations from the troposphere and, perhaps, the stratosphere were needed in order to specify the initial state with sufficient accuracy to reduce prediction errors due to uncertainties in the initial state of the atmosphere. When we add to these problems, that much had yet to be learned concerning numerical methods of integrating non-linear, coupled partial differential equations before one could attack such systems as those given in the earlier chapters, it is understandable that atmospheric models much simpler than those treated so far had to be invented for numerical prediction purposes.

We shall in this chapter consider a few of the models which have been used in practice in the field of numerical prediction. Some of these models are now of historical interest only, but they are nevertheless useful in discussing the structure of the atmosphere.

It is important to note that many, if not all, of the early prediction models were based on the vorticity equation and the selective use of the geostrophic approximation. We shall be concerned with these models here, but we cannot go into all the numerical problems in connexion with the finite difference form of the prediction equations.
The reasons for considering the vorticity equation and the geostrophic assumption will be taken up in the next section where we shall consider the so-called filter problem.

12.2 The Filter Problem

We have seen in Chapter XI that the general atmospheric equations permit solutions of many types: sound waves, internal and external gravity waves, inertia waves and Rossby waves. Synoptic experience and the understanding of the weather systems indicate that the various wave types mentioned above are of unequal importance in weather prediction. The motion systems which appear to carry the weather along as indicated by synoptic charts are the rather long waves (a few thousand kilometres in wavelength) which move rather slowly (say, 10 m s\(^{-1}\) as an order of magnitude) from west to east. The only wave type of those mentioned which satisfies this description is the Rossby wave which is characterized by a slow motion from west to east if the wavelength is sufficiently short. Of the other wave types we find that the sound waves and the gravity waves move too fast compared with the motion of the weather systems, while the inertia waves in a pure form hardly exist in the atmosphere because they require a vanishing pressure force. Based upon this discussion we come to the conclusion that it would be desirable to have a system of equations which describe the Rossby waves accurately, while the description of all the other wave types may be inaccurate or altogether missing. It is the design of such a new system of equations which defines the meteorological filter problem.

We found already in the treatment of the gravity waves in section 11.5 that we could eliminate the external gravity waves by adopting a modified lower boundary condition, i.e. \(\omega = 0, p = p_0\), replacing the more correct general condition. Changing the general condition at the lower boundary to \(\omega = 0\) is thus an example of a filter approximation whereby we exclude a physical phenomenon, in this case the external gravity waves, presumably because they are of little or no importance for the phenomena under investigation - the prediction of weather systems.

As another example we may mention the sound waves. If we wanted to exclude this wave type from the equations we could adopt the assumption of incompressibility, because a sound wave can exist only in a medium which is compressible. It is, however, conceivable that an assumption of incompressibility may be too drastic because compressibility may be important for some of the weather phenomena we wish to consider.

These examples show that the filter problem is a difficult problem, and that
it requires considerable meteorological insight to design a set of equations which are restricted in such a way that all unimportant phenomena are removed, while the essential features are left with only slight modifications. It is not even possible to know if a unique solution to such a problem exists, and it is not surprising that a systematic attack on the problem using the so-called scale analysis has been made. It is beyond the scope of this compendium to consider such a systematic attack on the problem, and we shall be satisfied by giving some examples of filter procedures.

We return for this purpose to the two-level model of the atmospheric flow. Let us consider a basic state of rest, i.e. \( \bar{u} = \bar{v} = \bar{\omega} = 0 \). Disregarding friction we find the following linearized equations of motion for the perturbations

\[
\frac{d\bar{u}_1}{dt} = -\frac{\partial \phi_1}{\partial x} + \nu \bar{v}_1
\]

\[
\frac{d\bar{v}_1}{dt} = -\frac{\partial \phi_1}{\partial y} - \nu \bar{u}_1
\]

\[
\frac{d\bar{u}_2}{dt} = -\frac{\partial \phi_2}{\partial x} + \nu \bar{v}_3
\]

\[
\frac{d\bar{v}_3}{dt} = -\frac{\partial \phi_3}{\partial y} - \nu \bar{u}_3
\]

(12.2.1)

where the subscripts as before refer to the 250 mb level (subscript 1) and the 750 mb level (subscript 3). For our purposes it is convenient to replace the four equations (12.2.1) by the vorticity and the divergence equations. Using the usual differentiation procedures we get:

\[
\frac{d\zeta_1}{dt} = -f D_1 - \beta \bar{v}_1
\]

\[
\frac{d\zeta_3}{dt} = -f D_3 - \beta \bar{v}_3
\]

(12.2.2)

\[
\frac{dD_1}{dt} = -2\nu \phi_1 + f \zeta_1 - \beta \bar{u}_1
\]

\[
\frac{dD_3}{dt} = -2\nu \phi_3 + f \zeta_3 - \beta \bar{u}_3
\]
where \( \zeta \) is the vorticity and \( D \) the divergence. We shall use the stream function \( \psi \) to express the vorticity, \( \zeta = \nabla^2 \psi \), and the velocity potential \( \chi \) to express the divergence, \( D = \nabla^2 \chi \). In terms of these two functions we have

\[
\begin{align*}
    u &= -\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x} \\
    v &= +\frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y}
\end{align*}
\]

(12.2.3)

We have thus six variables \( (\psi_1, \chi_1, \psi_1, \chi_1, \psi_2, \chi_2) \) entering the system (12.2.2). Using the boundary conditions \( \omega = 0 \) at \( p = 0 \) and \( p = p_4 \), of which the last condition eliminates the external gravity waves, we find from the continuity equation

\[
\begin{align*}
    D_1 - \frac{\omega^2}{P} &= 0 \\
    D_3 - \frac{\omega^2}{P} &= 0
\end{align*}
\]

(12.2.4)

It follows from (12.2.4) that \( D_1 + D_3 = 0 \). Denoting \( D_1 = D \) and therefore \( D_3 = -D \) we may combine the equations (12.2.4) in a single equation

\[
D + \frac{\omega^2}{P} = 0
\]

(12.2.5)

The result that the net divergence \( D_1 + D_3 \) vanishes leads us to replace the equations (12.2.2) and (12.2.4) by new equations obtained by addition and subtraction Denoting

\[
(\beta)_* = \frac{1}{2} \left[ (\beta)_1 + (\beta)_3 \right]
\]

(12.2.6)

and

\[
(\beta)_T = \frac{1}{2} \left[ (\beta)_1 - (\beta)_3 \right]
\]

(12.2.7)

we get
\[ \frac{d\xi_\star}{dt} + \beta v_\star = 0 \]
\[ 0 = -\nabla^2 \phi_\star + f\xi_\star - \beta u_\star \]  
(12.2.8)

\[ \nabla \cdot \mathbf{v}_\star = 0 \]

and

\[ \frac{\partial \xi_\star}{\partial t} + \beta v_T = \frac{f}{P} \omega_2 \]  
(12.2.9)

\[ \frac{\partial P}{\partial t} = -\nabla^2 \phi_T + f\xi_T - \beta u_T \]

\[ D + \frac{\omega_2}{P} = 0 \]

Finally we shall complete the system of equations by adding the thermodynamic equation

\[ \frac{d\phi_T}{dt} - \frac{\sigma_P}{2\omega_2} \omega_2 = 0 \]  
(12.2.10)

In order to keep the analysis as simple as possible we shall now assume that the perturbations are functions of \( x \) and \( t \) only, i.e.

\[ ( ) = ( ) e^{ik(x-ct)} \]  
(12.2.11)

Because of this assumption we have

\[ u = \frac{\partial x}{\partial x} \text{ and } v = \frac{\partial \psi}{\partial x} \]  
(12.2.12)
It is now obvious that the system (12.2.8) describes the vertical mean flow which according to the last equation is non-divergent, and it follows that $u_* = 0$. From the first equation we find in the usual way

$$c = -\frac{B}{k^2} = -c_R \tag{12.2.13}$$

indicating that the Rossby waves are present in the system.

From the system (12.2.9) supplemented by (12.2.10) we find, remembering that

$$f_T = \frac{\sigma^2\psi_T}{\partial x^2}, \quad D = \frac{\partial^2 x_T}{\partial x^2} \tag{12.2.14}$$

the following equations using (12.2.11):

$$ik(c + c_R)\hat{\psi}_T - f_0 \hat{\phi}_T = 0$$

$$f_0\hat{\phi}_T + ik(c + c_R)\hat{x}_T - \hat{\phi}_T = 0 \tag{12.2.15}$$

$$-\frac{\sigma^2 f^2}{2} k^2 \hat{x}_T - ikc \hat{\phi}_T = 0$$

The condition giving the equation for the phase speed is that the determinant of (12.2.15) vanishes in order to obtain non-trivial solutions. Denoting as before

$$c_I = \frac{f_0}{k}; \quad c_{IN} = \frac{a_2 f^2}{2} \tag{12.2.16}$$

we find, setting the determinant equal to zero, that

$$c(c + c_R)^2 - c_{IN}^2(c + c_R) - c_I^2 c = 0 \tag{12.2.17}$$
The solution to (12.2.17) will give the speed of the possible wave solutions in the model. It is seen that (12.2.17) is a third degree equation. We will have in general therefore three wave solutions. While (12.2.7) can be solved numerically without difficulty, it is more instructive to obtain a graphical solution. Let us define:

$$z = \frac{c}{c_{IN}}; \quad c_{R}^{*} = \frac{c_{R}}{c_{IN}}; \quad c_{I}^{*} = \frac{c_{I}}{c_{IN}} \quad (12.2.18)$$

(12.2.17) may now be written in the form

$$\left(z + c_{R}^{*}\right)^2 - c_{I}^{*2} = 1 + \frac{c_{R}}{z} \quad (12.2.19)$$

Defining

$$F(z) = \left(z + c_{R}^{*}\right)^2 - c_{I}^{*2} \quad (12.2.20)$$

and

$$G(z) = 1 + \frac{c_{R}}{z} \quad (12.2.21)$$

we find the solution to (12.2.17) by finding the intersections between the parabola $F(z)$ and the two branches of the hyperbola $G(z)$. An example is shown in Figure 12.1 where we have drawn the curves $F(z)$ and $G(z)$ for a wavelength of 4000 km. The three intersections are marked by circles. We notice that the two numerically large values are given approximately by the intersections of the parabola $F(z)$ with the asymptote $z = 1$. We find

$$c_{1,2} \approx -c_{R} \pm \sqrt{c_{IN}^2 + c_{I}^2} \quad (12.2.22)$$

The third root is small and negative. It is located between the value $-c_{R}$ and zero, and it can be found approximately by neglecting the first term in (12.2.17). We get
\[ c_3 \approx - \frac{c_{\text{IN}}^2}{c_{\text{IN}}^2 + c_1^2} c_R \]  

(12.2.23)

According to our earlier considerations of pure wave motion we find that \( c_{1,2} \) are slightly modified gravity-inertia waves, while \( c_3 \) is a slightly modified Rossby wave.

Having analysed the system in some detail we shall now show that the gravity-inertia waves \( c_{1,2} \) can be filtered out of the system by applying the geostrophic assumption in the following selective way: We replace the divergence equation in the system (12.2.9) by the geostrophic calculation, i.e.

\[ \xi_T = \frac{1}{f} \nabla^2 \phi_T \]  

(12.2.24)

which gives a direct relation between the stream function and the geopotential without involving the velocity potential. The equivalent of this is to neglect the term involving \( \xi_T \) in the middle equation of (12.2.15), and the frequency equation becomes

\[ c_1^2 c + c_{\text{IN}}^2 (c + c_R) = 0 \]  

(12.2.25)

or

\[ c = - \frac{c_{\text{IN}}^2}{c_1^2 + c_{\text{IN}}^2} c_R \]  

(12.2.26)

which coincides with the approximate value in (12.2.23).

The result obtained in this section is strictly true for the linearized perturbation equations. However, guided by these results we may make the hypothesis that if we use the geostrophic wind approximation in the selective way that it is used to compute the horizontal wind and the vorticity, but not the divergence, we will then have effectively filtered the internal gravity-inertia waves from the system. Numerical experiments with the non-linear equations show that this is the case. The resulting equations are the quasi-geostrophic equations. Including only the major terms these equations are

\[ \frac{\partial \xi}{\partial t} + \nabla \cdot \mathbf{v} (\xi + f) = f \frac{\partial \omega}{\partial p} \]
\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial p} \right) + \nabla \cdot \nabla \frac{\partial \phi}{\partial p} + \sigma \omega = 0
\]

(12.2.27)

where

\[
\zeta_g = \frac{1}{f} \nabla^2 \phi
\]

and

\[
\vec{v}_g = \frac{1}{f} \nabla \times \nabla \phi
\]

(12.2.29)

It is seen that (12.2.27) has only two dependent variables \( \phi \) and \( \omega \), and it is therefore a closed system, which may be characterized as the most simple baroclinic prediction system. While all undesirable wave types, according to experience, have been eliminated in (12.2.27), it is conceivable that the system is too simplified or too restrictive. The quasi-geostrophic equations have nevertheless been used extensively in analytical and numerical studies of the large-scale flow in the atmosphere.

12.3 Numerical Advantages

The system (12.2.7) (the quasi-geostrophic equations) is not only simpler than the unmodified equations as given for example by (3.3.7) or by (7.2.1) - (7.2.5), but there are also additional advantages in connexion with the numerical integrations of this system, and these advantages were probably originally one of the major reasons for introducing the quasi-geostrophic equations.

In order to understand how the quasi-geostrophic equations can give numerical advantages we must give a result of a mathematical nature. This result says that there is usually an upper limit to the time step which can be used in a numerical integration when the differential equation is replaced by a finite difference equation. The upper limit for the time step is given by the formula

\[
\Delta t \leq \frac{\Delta x}{c}
\]

(12.3.1)

where \( \Delta x \) is a grid size and \( c \) is the maximum speed of propagation within the system. We shall not give any proof of (12.3.1), but just mention that it is known as the Friedrich-Courant-Levy criterion.
According to (12.3.1) we can find the largest possible time step when we know the smallest grid size and the largest possible value of c. To illustrate the application of (12.3.1) let us assume that we were to integrate the original equations without any simplifications. The smallest time step would then be in the vertical direction, say \( z = 2 \text{ km} \) as originally proposed by Richardson, and the largest speed of propagation would be about 330 m s\(^{-1}\). We would then have

\[
\Delta t \leq 6 \text{ s} \tag{12.3.2}
\]

A time step as short as that indicated by (12.3.2) would be prohibitive in large-scale predictions.

Let us next assume that we have introduced the hydrostatic assumption in our equations. The effect of this modification is to filter the vertical component of the sound waves, although sound waves may still propagate in the horizontal directions. We state this result without proof, but the main effect is that \( \Delta x \) should now be interpreted as the horizontal grid size, and we find

\[
\Delta t \leq 1155 \text{ s} \tag{12.3.3}
\]

when we use \( \Delta x = 301 \text{ km} \) and \( c = 330 \text{ m s}^{-1} \). The effect is thus to increase the time step to about 20 min as a result of introducing the hydrostatic equation. The time step given by (12.3.3) gives an order of magnitude of the time step which should be used in connexion with the equations described in Chapters III and VII.

Let us turn finally to the quasi-geostrophic equations derived in this chapter. As has been explained in section 12.2, we have in these equations filtered all fast moving waves from the equations. The maximum speed of propagation is then of the order of magnitude of the wind speed, and assuming that \( c \leq 100 \text{ m s}^{-1} \), we find

\[
\Delta t \leq 3810 \text{ s} \approx 1 \text{ hour} \tag{12.3.4}
\]

which shows that a considerable advantage has been obtained by filtering the fast moving waves.
12.4  The Equivalent Barotropic Model

One of the early models which has played a major role in practical numerical prediction is the barotropic model usually applied at 500 mb. The governing equation for the barotropic model is

\[
\frac{d(\zeta + f)}{dt} = \frac{\partial \zeta}{\partial t} + \nabla \cdot \mathbf{v} (\zeta + f) = 0 \tag{12.4.1}
\]

and the requirement for its validity is that the atmosphere should move horizontally without divergence. It is not immediately obvious that (12.4.1) is applicable to the real atmosphere at all, or, if it is applicable in an approximate way, that it should be used at 500 mb.

In order to justify the use of (12.4.1) at 500 mb it is customary to consider the equivalent barotropic model which is defined as a model atmosphere in which the vertical variation of the (geostrophic) horizontal wind is given by the relation

\[
\mathbf{v}(x,y,p,t) = A(p) \mathbf{v}_M(x,y,t) \tag{12.4.2}
\]

where \(A(p)\) is an empirical function of pressure describing the vertical variation of the wind speed and where the wind \(\mathbf{v}_M\) is the vertically averaged wind, i.e.

\[
\mathbf{v}_M = \frac{1}{p_0} \int_0^{p_0} \mathbf{v} \, dp. \tag{12.4.3}
\]

We note from (12.4.2) and (12.4.3) that

\[(A(p))_M = 1 \tag{12.4.4}\]

i.e. the vertical average of \(A(p)\) is unity.
The implication of $A(p)$ is that while the wind may change in speed with height, it will always have the same direction, namely the direction of $\mathbf{v}_M$. We may also express the same statement in another way. The thermal wind is

$$\frac{d\mathbf{v}}{dp} = \frac{dA}{dp} \mathbf{v}_M$$  \hspace{1cm} (12.4.5)

Now, the thermal wind has thus the same direction as $\mathbf{v}_M$, but it is also directed along the isotherms if we assume geostrophy. It follows therefore that the isotherms at any level are parallel to the contours, or that the temperature advection vanishes in an equivalent barotropic atmosphere.

Let us now consider the vorticity equation

$$\frac{d\zeta}{dt} + \mathbf{v} \cdot \nabla (\zeta + f) = f \frac{d\omega}{dp}$$  \hspace{1cm} (12.4.6)

with the simplified boundary conditions $\omega = 0$, $p = 0$ and $p = p_0$. We note that it follows from (12.4.2) that

$$\zeta = A(p)\zeta_M$$  \hspace{1cm} (12.4.7)

Introducing (12.4.2) and (12.4.7) in (12.4.6) and integrating through the whole atmosphere, we find

$$\frac{d\zeta_M}{dt} + (A^2) \mathbf{v}_M \cdot \nabla \zeta_M + \mathbf{v}_M \cdot \nabla f = 0$$  \hspace{1cm} (12.4.8)

which is the vorticity equation for the vertical mean flow.
The shape of \( A(p) \) is given schematically in Figure 12.2, where \( A(p) \) is the full curve with its vertical average of 1. The quantity entering (12.4.8) is the vertical average of the function \( \Lambda^2 \), which is indicated on Figure 12.2 by a dashed curve, whose vertical average is \( (\Lambda^2)_M \). We define now a level \( p_* \) such that

\[
A_* = A(p_*) = (\Lambda^2)_M
\]

(12.4.9)

We have then

\[
\xi_* = A_* \xi_M, \quad \vec{v}_* = A_* \vec{v}_M
\]

(12.4.10)

which, when introduced in (12.4.8) and combined with (12.4.9), leads to

\[
\frac{1}{A_*} \frac{\xi_*}{t_*} \frac{(\Lambda^2)_M}{A_*} \vec{v}_* \cdot \nabla \xi_* + \frac{1}{A_*} \vec{v}_* \cdot \nabla \phi = 0
\]

(12.4.11)
\[ \frac{\partial \kappa}{\partial t} + \nabla \cdot \nu (\kappa + f) = 0 \] (12.4.12)

This equation shows that the equivalent barotropic model atmosphere behaves as if it were barotropic at the level \( p_0 \), which therefore is called the equivalent barotropic level. We notice that this level is located somewhat higher in the atmosphere than the level where the wind is equal to the mean wind (see Figure 12.2, where the latter level is denoted \( p_M \)). It is found in practice that \( p_M \) is located around 600 mb while \( p_0 \) is located at 500 mb, approximately. The position of the equivalent barotropic level is in reality a function of location and time, although it is assumed to be located at a constant pressure in each application. The discrepancy may explain some of the errors in the use of the equivalent barotropic forecasts.

We emphasize that (12.4.12) applies only at the equivalent barotropic levels. It is thus possible to have a vertical velocity in an equivalent barotropic atmosphere. In order to investigate the behaviour of the vertical velocity we introduce (12.4.7) in (12.4.6) and obtain

\[ \frac{\partial \kappa}{\partial t} + \nabla \cdot \nu (\kappa + f) = 0 \] (12.4.13)

When we multiply (12.4.8) by \( A(p) \) and subtract the resulting equation from (12.4.13) we obtain

\[ \frac{\partial \omega}{\partial p} = (A^2 - (A^2)_M) \frac{1}{F} \nabla \cdot \nu \kappa \] (12.4.14)

or

\[ \omega = c(p) \frac{1}{F} \nabla \cdot \nu \kappa \] (12.4.15)

where

\[ c(p) = \int_0^p (A^2 - (A^2)_M) dp \] (12.4.16)

We observe that \( c(p_0) = 0 \) such that the lower boundary condition is satisfied. We note further that \( \omega \) is proportional to the vorticity advection, \( \nabla_M \cdot \nu \kappa_M \). If we introduce a local co-ordinate system in which the \( s \)-axis is along the streamlines,
while the \( n \)-axis is along the normal to the streamline, we find

\[
\nabla M \cdot \nabla \xi_M = \frac{V}{M} \frac{\partial \xi_M}{\partial s} \tag{12.4.17}
\]

and it is seen that \( \omega \) is positive (sinking motion) when the wind blows from low to high values of the vorticity, while \( \omega \) is negative in the opposite case. As a special result we find that we will have sinking motion between the ridge and the trough looking downstream, while rising motion is found between the trough and the ridge.

The whole reasoning above is true only when \( c = c(p) > 0 \) while the opposite conditions hold when \( c < 0 \). It is therefore of some importance to investigate when each of these conditions hold. In order to get an understanding of this it is most convenient to consider a comparatively simple example. Let

\[
A = A(p) = \begin{cases} 
2 \frac{p}{p_T}; & 0 \leq p \leq p_T; \quad p_T = 250 \text{ mb} \\
2 \frac{p_T - p}{p_o - p_T}; & p_T \leq p \leq p_o; \quad p_o = 1000 \text{ mb}
\end{cases} \tag{12.4.18}
\]

It is easily seen that \( (A^2)_M = \frac{4}{3} \), and that

\[
c(p) = \begin{cases} 
- \frac{4}{3} \frac{p^2}{p_T} (p_T - p); & 0 \leq p \leq p_T \\
\frac{4}{3} \left( \frac{p_o - p}{p_o - p_T} \right)^2 (p - p_T); & p_T \leq p \leq p_o
\end{cases} \tag{12.4.19}
\]

The function \( A(p) \) in (12.4.13) is a simple linear function which, however, is realistic in the sense that it has a maximum at 250 mb simulating the maximum (jet-stream) in a real atmosphere. We note incidentally that the tropospheric level when the wind is equal to the mean wind, i.e. \( A = 1 \) is found at \( p_T = 625 \text{ mb} \) while the tropospheric equivalent barotropic level is at \( p_T = 500 \text{ mb} \).

It is furthermore seen that \( c < 0 \) when \( p < p_T \) (the stratosphere) while \( c > 0 \) when \( p_T < p < p_o \). The zero points for \( c \) are found at \( p = 0, p = p_T \), and \( p = p_o \), while the non-divergent levels, i.e. levels when \( \partial \omega / \partial p = 0 \), are found where

\[
A^2 - (A^2)_M = 0 \tag{12.4.20}
\]

or

\[
A = (A^2)_M \tag{12.4.21}
\]
It is thus seen that the equivalent barotropic level is also a level of non-divergence as one would expect from (12.4.12).

The functions $A(p)$ and $c(p)$ for the example above are shown in Figure 12.3, where $c(p)$ has been normalized in such a way that the tropospheric maximum is unity. The profile $c_N(p)$ shows clearly the opposite signs below and above the wind maximum and the existence of the maximum in the middle of the atmosphere giving convergence above and divergence below or vice versa depending on the sign of the vorticity advection.

The properties of the equivalent barotropic model are the main reasons for the application of the barotropic vorticity equation at 500 mb. Such barotropic forecasts are made routinely by all major numerical forecast centres in the world for periods of up to 72 hours. The very fact that one continues to produce barotropic forecasts many years after the model was first introduced speaks for the fact that these forecasts, in spite of the simplicity of the model, describe a major part of the changes in the large-scale atmospheric flow.
12.5 Quasi-geostrophic Baroclinic Models

The models are based on the system (12.2.27). The two dependent variables are the geopotential $\Phi$ and the vertical velocity $\omega$. Since the initial data consist of the geopotential of the isobaric surfaces, it is most convenient to eliminate the vertical velocity $\omega$ from the system. Considering as before the case of adiabatic and frictionless flow we find from the thermodynamic equation

$$\omega = - \left( \frac{\partial}{\partial t} \left[ \frac{1}{\sigma} \frac{\partial \Phi}{\partial p} \right] + \nabla \cdot \nabla \left[ \frac{1}{\sigma} \frac{\partial \Phi}{\partial p} \right] \right)$$

(12.5.1)

and by differentiation

$$\frac{\partial \omega}{\partial p} = \left( \frac{\partial}{\partial p} \left[ \frac{1}{\sigma} \frac{\partial \Phi}{\partial p} \right] + \nabla \cdot \nabla \left[ \frac{1}{\sigma} \frac{\partial \Phi}{\partial p} \right] \right)$$

(12.5.2)

where we note that the term

$$\frac{1}{\sigma} \frac{\partial \nabla \cdot \left( \frac{\Phi}{\partial p} \right)}{\partial p} = 0$$

(12.5.3)

because of the thermal wind equation. The expression (12.5.2) is substituted into the vorticity equation in (12.2.27), and we find assuming $f = f_0$, where $f_0$ is a standard value, in the divergence term

$$\frac{\partial}{\partial t} \left[ \zeta_g + \frac{\partial}{\partial p} \left( \frac{f_0}{\sigma} \frac{\partial \Phi}{\partial p} \right) \right] + \nabla \cdot \nabla \left[ \zeta_g + f + \frac{\partial}{\partial p} \left( \frac{f_0}{\sigma} \frac{\partial \Phi}{\partial p} \right) \right] = 0$$

(12.5.4)

which shows that the term

$$\xi = \zeta_g + f + \frac{\partial}{\partial p} \left( \frac{f_0}{\sigma} \frac{\partial \Phi}{\partial p} \right)$$

(12.5.5)

is conserved in the horizontal geostrophic flow. The expression in (12.5.5) is called the quasi-geostrophic potential vorticity, because it is the conservative quantity in quasi-geostrophic flow. $\xi$ can only be changed by heating and friction in geostrophic flow.

The quasi-geostrophic baroclinic models could be based on (12.5.4) which has the geopotential as the only dependent variable, but the direct application of (12.5.4) is difficult because the boundary conditions are most conveniently expressed
in \( \omega \). The finite difference equations are therefore normally formed using the system (12.2.27) directly.

\[
\begin{align*}
\omega &= 0 & 0 \\
\Phi &= 1 \\
\omega_2 &= 2 \\
\Phi &= 2n-1 \\
\omega &= 2n \\
\Phi &= 2n+1 \\
\omega &= 2n+2 \\
\Phi &= 2n+3 \\
\omega &= 2N-2 \\
\Phi &= 2N-1 \\
\omega &= 0 & 2N
\end{align*}
\]

Figure 12.4

We shall demonstrate the main points in the finite difference system in the vertical direction as illustrated in Figure 12.4. The boundary conditions will be \( \omega = 0 \) at \( p = 0 \) and \( p = p_0 \). It is now customary to divide the whole depth of the atmosphere in equal parts. Numbering the levels: 0, 1, ..., 2N-1, 2N we shall have the vertical velocity at the even levels including \( \omega_0 = 0, \omega_{2N} = 0 \), while the geopotential will be at the odd levels. We denote the pressure interval between two even (or two odd) levels by \( \Delta p \). We have

\[
\Delta p = \frac{p_0}{N}
\]  

(12.5.6)
The number $N$ is used to name the model as an $N$-level baroclinic model, because the geopotential is available at $N$ levels.

At an arbitrary odd level $(2n+1)$ we have then:

$$\frac{\partial \zeta_{2n+1}}{\partial t} + \nabla \cdot \mathbf{v}_{2n+1} = f \frac{\omega_{2n+1}}{\Delta \varphi}$$  \hspace{1cm} (12.5.7)

Applying the thermodynamic equation at levels $2n+2$ and $2n$ we get:

$$+ \frac{1}{\Delta \varphi} \left[ \frac{\partial (\phi_{2n+3} - \phi_{2n+1})}{\partial t} \right] + \nabla \cdot \mathbf{v}_{2n+2} (\phi_{2n+3} - \phi_{2n+1}) + \sigma_{2n+2} \omega_{2n+2} = 0$$  \hspace{1cm} (12.5.8)

and

$$+ \frac{1}{\Delta \varphi} \left[ \frac{\partial (\phi_{2n+1} - \phi_{2n-1})}{\partial t} \right] + \nabla \cdot \mathbf{v}_{2n} (\phi_{2n+1} - \phi_{2n-1}) + \sigma_{2n} \omega_{2n} = 0$$  \hspace{1cm} (12.5.9)

It is now seen that (12.5.8) can be solved for $\omega_{2n+2}$ and (12.5.9) for $\omega_{2n}$ and these expressions can be substituted in (12.5.7), and we have then $N$ equations of the type (12.5.7) with the unknowns $\phi_1$, $\phi_3$, ..., $\phi_{2N-1}$. Two of these equations are special, i.e. the equations for $\zeta_1$ and $\zeta_{2N-1}$, because the boundary value $\omega_0 = 0$ appears in the first equation, while $\omega_{2N} = 0$ appears in the last. These equations are therefore simpler. We note further that there appears to be a formal difficulty in (12.5.8) and (12.5.9) because we need the horizontal wind at the even levels where it is unavailable. This difficulty is overcome by defining the wind at an even level as the average of the winds above and below, i.e.

$$\mathbf{v}_{2n} = \frac{1}{2}(\mathbf{v}_{2n-1} + \mathbf{v}_{2n+1}) = \mathbf{v}_{2n+1} + \frac{1}{2}(\mathbf{v}_{2n-1} - \mathbf{v}_{2n+1})$$  \hspace{1cm} (12.5.10)

and

$$\mathbf{v}_{2n+2} = \frac{1}{2}(\mathbf{v}_{2n+1} + \mathbf{v}_{2n+3}) = \mathbf{v}_{2n+1} - \frac{1}{2}(\mathbf{v}_{2n+1} - \mathbf{v}_{2n+3})$$  \hspace{1cm} (12.5.11)

Substituting the expressions (12.5.10) and (12.5.11) in (12.5.8) and (12.5.9), respectively, we find using the thermal wind relation that

$$\omega_{2n+2} = \frac{1}{\sigma_{2n+2} \Delta \varphi} \left[ \frac{\partial (\phi_{2n+1} - \phi_{2n+3})}{\partial t} \right] + \nabla \cdot \mathbf{v}_{2n+1} (\phi_{2n+1} - \phi_{2n+3})$$  \hspace{1cm} (12.5.12)
and

$$\omega_{2n} = \frac{1}{\sigma_{2n} \Delta P} \left[ \frac{\partial (\phi_{2n-1} - \phi_{2n+1})}{\partial t} + \nabla \cdot \nabla (\phi_{2n-1} - \phi_{2n+1}) \right]$$  \hspace{1cm} (12.5.13)$$

and we can finally write (12.5.7) in the form

$$\frac{\partial \xi_{2n+1}}{\partial t} + \nabla \phi_{2n+1} \cdot \nabla \phi_{2n+1} = 0$$  \hspace{1cm} (12.5.14)$$

where

$$\xi_{2n+1} = f + \xi_{2n+1} + \frac{f}{\Delta P} \left[ \frac{\phi_{2n-1} - \phi_{2n+1}}{\sigma_{2n}} - \frac{\phi_{2n+1} - \phi_{2n+3}}{\sigma_{2n+2}} \right]$$  \hspace{1cm} (12.5.15)$$

or

$$\xi_{2n+1} = f + \xi_{2n+1} + q_{2n}^2 \left( \phi_{2n-1} - \phi_{2n+1} \right) - q_{2n+2}^2 \left( \phi_{2n+1} - \phi_{2n+3} \right)$$  \hspace{1cm} (12.5.16)$$

where

$$q_{2n}^2 = \frac{f}{\sigma_{2n}(\Delta P)^2}$$  \hspace{1cm} (12.5.17)$$

It is thus seen that we may incorporate the boundary conditions in (12.5.16) by setting $q_0^2 = 0$ and $q_{2N}^2 = 0$. Equation (12.5.14) is used to advance the quasi-

giostrophic potential vorticity $\xi$ a time step. Equation (12.5.16) where we know

the left side, is used to obtain values of $\phi$ in all the grid points at all the

levels. The solution of (12.5.16) for the geopotentials must be done by a numerical

trial and error method known as the relaxation method, but it is beyond the scope of

this treatment to go into the details of numerical procedures.

The vertical velocities were eliminated from the system using (12.5.17) and

(12.5.13). It is important to know that when we have found the new values of the

geopotentials, we can go back to (12.5.12), (12.5.13) and obtain the new values of the

vertical velocities in all the grid points.

We note finally that the two-level model is the most elementary application

of the general scheme given above. In this case we have the geopotential at the

levels 1 and 3, while $\omega_2$ is the only vertical velocity apart from the boundary con-
ditions. We have thus from (12.5.12) and (12.5.13)

$$\omega_2 = \frac{1}{\sigma_2 \Delta P} \left[ \frac{\partial (\phi_1 - \phi_3)}{\partial t} + \nabla \cdot \nabla (\phi_1 - \phi_3) \right]$$  \hspace{1cm} (12.5.13)$$
and
\[ \omega_2 = \frac{1}{c_2 \Delta P} \left[ \frac{\partial \phi_1 - \phi_3}{\partial t} + \nabla \cdot \nabla (\phi_1 - \phi_3) \right] \] (12.5.19)

while (12.5.14) leads to the following equations
\[ \frac{\partial \xi_1}{\partial t} + \nabla_1 \cdot \nabla \xi_1 = 0 \] (12.5.20)
\[ \frac{\partial \xi_3}{\partial t} + \nabla_3 \cdot \nabla \xi_3 = 0 \] (12.5.21)

and
\[ \xi_1 = f + \xi_1 - q_2 \phi_1 - \phi_3 \] (12.5.22)
\[ \xi_3 = f + \xi_3 + q_2 \phi_1 - \phi_3 \] (12.5.23)

12.6 The \( \omega \)-Equation

The emphasis in the last section was on the prediction of the geopotential using a quasi-geostrophic baroclinic model. In many investigations it is of interest to obtain a calculation of the vertical velocity at a given synoptic time. Such information can be obtained by solving the so-called \( \omega \)-equation which will now be derived. We start from (12.2.27) which we write in the form
\[ \frac{\partial \nabla_1 \phi}{\partial t} + \nabla_1 \cdot \nabla (\nabla_1 f + f) = f^2 \frac{\partial \omega}{\partial \rho} \]
\[ \frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial \rho} \right) + \nabla \cdot \nabla \left( \frac{\partial \phi}{\partial \rho} \right) = -\rho \omega \] (12.6.1)

The first of these equations is differentiated with respect to pressure, while we take the Laplacian of the second. Subtraction of the resulting equations gives
\[ f^2 \frac{\partial^2 \omega}{\partial \rho^2} + \sigma \nabla^2 \omega = f \frac{\partial}{\partial \rho} \left[ \nabla \cdot \nabla (f + \xi) \right] - \nabla^2 \left[ \nabla \cdot \nabla \frac{\partial \phi}{\partial \rho} \right] \] (12.6.2)

which is the \( \omega \)-equation. It is of a diagnostic nature containing no time derivatives. The left-hand side of the equation is a linear second order operator, while the right-hand side can be calculated knowing the geopotential at various pressure levels in
the vertical direction (see Figure 12.4). Knowing the right side it is possible to solve (12.6.2) for \( \omega \), specifying lateral and vertical boundary conditions, by a numerical relaxation technique. The finite difference formulation of (12.6.2) is in fact very similar to (12.5.16).

We shall first apply (12.6.2) to the two-level model, and we shall consider a simple baroclinic wave. Using the same notations as before we find

\[
\omega - \frac{2f_0^2}{\sigma^2} \omega = -\frac{1}{P} \left[ \nabla_1 \cdot \nabla (\zeta_1 + f) - \nabla_3 \cdot \nabla (\zeta_3 + f) \right] + \frac{2}{P} \nabla^2 \left[ \nabla_1 \cdot \nabla \phi_T \right] \tag{12.6.3}
\]

Denoting

\[
\nabla_1 = \nabla_M + \nabla_T, \quad \zeta_1 = \zeta_M + \zeta_T \\
\nabla_3 = \nabla_M - \nabla_T, \quad \zeta_3 = \zeta_M - \zeta_T \quad q = \frac{2f_0^2}{\sigma^2 P^2} \tag{12.6.4}
\]

we find

\[
\omega = \frac{2f_0}{\sigma^2 P} \left[ \frac{1}{P} \nabla_1^2 \left( \nabla_M \cdot \nabla \phi_T \right) - \nabla_M \cdot \nabla \zeta_M - \nabla_T \cdot \nabla \zeta_T - \beta \nabla_T \right] \quad \tag{12.6.5}
\]

which is the \( \omega \)-equation for the two-level model.

We shall next solve (12.6.5) using a simple wave pattern given by

\[
\phi_M = -f_0 U_M y + A_M \sin kx \\
\phi_T = -f_0 U_T y + A_T \sin (kx + \alpha_T) \tag{12.6.6}
\]

\[<\alpha>T> 0 \quad \alpha_T > 0 \quad x = L \quad x = 0 \quad L \]

\[\text{Figure 12.5}\]
The wave patterns given by (12.6.6) are shown schematically in Figure 12.5. \( U_M \) and \( U_T \) are constant zonal winds, \( A_M \) and \( A_T \) are the amplitudes of the two waves which have the wavelength \( L = 2\pi/k \). The phase angle \( \alpha_T \) measures the difference between the maxima of the two waves where \( \alpha_T > 0 \) is used in the construction of Figure 12.5. The situation depicted in the figure is normal for waves in the middle latitudes.

When (12.6.6) is substituted in (12.6.5) we obtain after some elementary calculations that

\[
v^2 \omega_2 - q^2 \omega_2 = \frac{P}{fo} q^2 k^2 \left[ 2U_T v_M - c_R v_T \right]
\]

(12.6.7)

where

\[
v_M = \frac{k}{fo} A_M \cos kx = \Phi_M \cos kx
\]

(12.6.8)

and

\[
v_T = \frac{k}{fo} A_T \cos(kx + \alpha_T) = \Phi_T \cos(kx + \alpha_T)
\]

(12.6.9)

It is now easy to see that \( \omega_2 \) must have the form

\[
\omega_2 = -C \cos kx + D \cos (kx + \alpha_T)
\]

(12.6.10)

where \( C \) and \( D \) can be determined by a substitution of (12.6.10) in (12.6.7). We find

\[
C = \frac{P}{fo} \frac{q^2 k^2}{q^2 + k^2} 2U_T \Phi_M
\]

\[
D = \frac{P}{fo} \frac{q^2 k^2}{q^2 + k^2} c_R \Phi_T
\]

(12.6.11)
Figure 12.6 shows the same wave arrangement as in Figure 12.5. It is seen from (12.6.7) combined with (12.6.10) and (12.6.11) that the first term on the right-hand side will give rising motion between the trough $T_M$ and the ridge $R_M$ in the mean flow, while sinking motion due to the same term is found between $R_M$ and $T_M$ (see arrows on the figure). On the other hand, the last term depending on $c_R$ will result in sinking motion between the trough $T_T$ and the ridge $R_T$ in the thermal flow and rising motion between $R_T$ and $T_T$ looking downstream (see dashed arrows on Figure 12.6). We can thus conclude that as long as the thermal wave is lagging behind the mean wave, we will have rising motion just behind the trough $T_M$—in fact between $T_T$ and $T_M$, while we will have rising motion between $R_T$ and $R_M$, because the two terms in (12.6.7) work in the same direction in the two locations. In the remaining locations ($R_M$ to $T_T$, and $T_M$ to $R_T$) we will have opposite contributions from the two terms. However, as long as the wavelength is small, i.e. $c_R$ small, we will have dominance of the first term and thus rising motion between $T_M$ and $R_M$, essentially, and sinking motion between $R_M$ and $T_M$.

The situation for long waves may be different because $c_R$ increases as the square of the wavelength, and it is possible that the second term may dominate. We can get an estimate of when this happens by considering the case of $q_T = 0$. Assuming that $\Phi_M = \Phi_T$ we find that the critical wavelength is determined by

$$2U_T - c_R = 0$$ (12.6.12)

If $U_T = 10 \text{ m s}^{-1}$ and $\beta = 16 \times 10^{-12} \text{ m}^{-1} \text{ s}^{-1}$, we find that the wavelength must be of the order of 7000 km if (12.6.12) should be fulfilled. We conclude therefore that the beta-effect is of minor importance for the determination of the vertical velocity as long as we restrict ourselves to reasonably short waves. In the following considerations we shall completely disregard the variation of $f$.

We return now to the general $\omega$-equation (12.6.2), and let us first consider the left-hand side of the equation. A general component which would satisfy the upper and lower boundary condition, i.e. $\omega = 0$, $p = 0$ and $p = p_o$, can be written in the form

$$\omega(x,y,p) = A(\omega) \sin \left( \pi \frac{P}{P_o} \right) \sin k_1 x \sin k_2 y$$ (12.6.13)

where $k_1$ and $k_2$ are wave numbers in the $x$- and $y$- directions, respectively. When we evaluate the left-hand side of (12.6.2) we find

$$f_o^2 \frac{\partial^2 \omega}{\partial p^2} + \sigma \nabla^2 \omega = - \left( f_o^2 \frac{n^2 \Pi^2}{P_o^2} + k_1^2 + k_2^2 \right) \omega$$ (12.6.14)
and it is thus seen that the operator on $\omega$ is such that the left-hand side of (12.6.2) must be approximately proportional to $-\omega$. We may thus write (12.6.2) in the form

$$\quad -C_\omega = F \quad (12.6.15)$$

where $F$ is a notation for the complete right-hand side. The major problem is now to evaluate $F$. With $f = f_0$, we find that

$$F = -\frac{1}{f_0} \left[ \frac{\partial}{\partial p} \right] \left\{ J(\nabla^2 \phi, \phi) \right\} - \nabla^2 \left\{ J\left( \frac{\partial \phi}{\partial p}, \phi \right) \right\} \quad (12.6.16)$$

where the Jacobian $J(a, \beta)$ is defined

$$J(a, \beta) = \frac{\partial a}{\partial x} \frac{\partial \beta}{\partial y} - \frac{\partial a}{\partial y} \frac{\partial \beta}{\partial x} \quad (12.6.17)$$

It is easy to show that

$$\frac{\partial}{\partial p} \left[ J(a, \beta) \right] = J\left( \frac{\partial a}{\partial p}, \beta \right) + J(a, \frac{\partial \beta}{\partial p}) \quad (12.6.18)$$

It is more laborious, but straightforward, to prove the following relation

$$\nabla^2 J(a, \beta) = J(\nabla^2 a, \beta) + J(a, \nabla^2 \beta) - 2\Lambda(a, \beta) \quad (12.6.19)$$

where

$$\Lambda(a, \beta) = \frac{\partial^2 a}{\partial x \partial y} \left( \frac{\partial^2 \beta}{\partial x^2} - \frac{\partial^2 \beta}{\partial y^2} \right) - \frac{\partial^2 \beta}{\partial x \partial y} \left( \frac{\partial^2 a}{\partial x^2} - \frac{\partial^2 a}{\partial y^2} \right) \quad (12.6.20)$$

Using the relations (12.6.18) and (12.6.19) we find that

$$F = -\frac{1}{f_0} \left[ J(\nabla^2 \phi, \phi) + J(\nabla^2 \phi, \frac{\partial \phi}{\partial p}) - J(\nabla^2 (\frac{\partial \phi}{\partial p}), \phi) - J(\frac{\partial \phi}{\partial p}, \nabla^2 \phi) \right. \nonumber$$

$$+ \left. 2\Lambda(\frac{\partial \phi}{\partial p}, \phi) \right] \quad (12.6.21)$$

or, using the hydrostatic relation and recalling that $J(a, \beta) = -J(\beta, a)$,

$$F = \frac{2R}{f_0} \left[ J(\nabla^2 \phi, T) + \Lambda(T, \phi) \right] \quad (12.6.22)$$
Combining (12.6.15) and (12.6.22) and recalling furthermore that \( \omega = - g \rho w \), approximately, we find

\[
w \approx c_* \left[ J(\nabla^2 \phi, T) + \Lambda(T, \phi) \right] \tag{12.6.23}\]

where \( c_* > 0 \) is a constant combining \( c \) in (12.6.15) and the coefficient appearing in (12.6.22).

The function \( \Lambda(T, \phi) \) is called the deformation function because the differential combinations entering the definition (12.6.20) are related to the geostrophic deformations. The two components of the deformation are

\[
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = - \frac{1}{f_o} \frac{\partial^2 \phi}{\partial x \partial y} \tag{12.6.24}
\]

and

\[
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{f_o} \left( \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} \right) \tag{12.6.25}
\]

and it is seen from (12.6.20) that these combinations are exactly those entering the definition of \( \Lambda(T, \phi) \). The expressions in \( T \) entering \( \Lambda(T, \phi) \) are then the deformation components of the geostrophic thermal wind.

Test calculations using observed data seem to indicate that

\[
|\Lambda(T, \phi)| < |J(\nabla^2 \phi, T)| \tag{12.6.26}
\]

and we have thus approximately

\[
w \approx c_* J(\nabla^2 \phi, T) \tag{12.6.27}
\]

which shows that \( w \) is proportional to the advection of vorticity by the thermal wind.

We may see this by introducing a local co-ordinate system with \( s \) along the thermal wind and \( n \) normal to \( s \) and pointing towards colder air (see Figure 12.7).
We have then \( \frac{dT}{ds} = 0 \), and we get

\[
w \approx c_s \frac{\partial^2 \Phi}{\partial s \partial n} \frac{dT}{dn}
\]  
(12.6.28)

Since \( \frac{dT}{dn} < 0 \) by definition, we find that

\( w > 0 \), if \( \frac{\partial^2 \Phi}{\partial s} < 0 \)

and

\( w < 0 \), if \( \frac{\partial^2 \Phi}{\partial s} > 0 \)

or

**Rising motion is found when the thermal wind blows from higher to lower values of the vorticity, and sinking motion in the opposite case.**

The rule just derived may naturally also be used to determine the distribution of the vertical velocity in a typical upper-air atmospheric wave, with essentially the same result. We shall in this section apply the rule to a typical low-level wave cyclone as shown in Figure 12.8. Since the maximum vorticity is close to the centre and decreasing as we move away along the isotherms we find, according to the rule, rising motion in point A, but sinking motion in point B. If we think of the flow in Figure 12.8 as a frontal wave, we have deduced the rising motion ahead of the warm front and the sinking motion behind the cold front without thinking of these surfaces as discontinuity surfaces.
Figure 12.3

The material in this section has been included to provide an understanding of the field of vertical motion and the processes responsible for creating rising and sinking motion. The emphasis has been on a qualitative understanding. If a detailed quantitative picture is needed, one must solve equation (12.6.2) by numerical methods.
Chapter XIII

ATMOSPHERIC TURBULENCE

13.1 Introduction

The atmosphere has motion on many scales in space and time. On the space scale the motion varies from the largest scale, comparable to the dimensions of the earth ($10^4$ km), through Rossby and cyclone waves ($5 \times 10^3$ km - $10^3$ km), mesoscale motion ($10^2$ - $10^1$ km), and convection ($10^1$ - 1 km) to small local systems and vortices, of an order of magnitude measured in cm or even mm. We may therefore say that the spectrum of atmospheric motion is very wide ranging over several orders of magnitude. A similar variation in time is found. Apart from the annual or even longer fluctuations we have a time scale of weeks for the largest space scale, over a few days for the Rossby and cyclone waves to hours, minutes and seconds for the smaller scale motion.

It is impossible to observe atmospheric motion on all these time and space scales simultaneously as it is impossible to observe the three dimensional distributions of pressure, temperature and humidity in such fine detail that all scales can be represented. For any given density of observations there is a smallest scale of motion (admittedly ill-defined) which can be represented by the data in an accurate way. Whatever may exist on an even smaller scale cannot be treated in any detail.

The main problem now is whether we are permitted to disregard the small scale (unobservable) motion altogether, or whether we must somehow pay attention to it. If the latter is true, we can only treat the small-scale motion by relating it to the large-scale motion. Statistical methods are ideal tools for this purpose. The very nature of the laws governing the atmosphere makes it likely that interaction takes place between phenomena on different scales. The reasons for this are the nonlinear terms in the equations. In order to illustrate, by an example, such a nonlinear interaction we shall consider one of the nonlinear terms, say $u(\partial u/\partial x)$. We can, in general, assume that $u = u(x)$ contains many scales. We shall consider only two scales with wave numbers $k_1$ and $k_2$, and we shall write for each of the components:

$$\begin{align*}
    u_1 &= A_1 \cos k_1 x + B_1 \sin k_1 x \\
    u_2 &= A_2 \cos k_2 x + B_2 \sin k_2 x
\end{align*}$$

(13.1.1)
One part of the interaction between the two components is \( u_1 \frac{\partial u_2}{\partial x} \). When (13.1.1) is substituted in this expression and the result reduced using trigonometric relations we find

\[
\begin{align*}
 u_1 \frac{\partial u_2}{\partial x} &= \frac{1}{2} k_2 (A_1 B_2 + A_2 B_1) \cos(k_1 + k_2) x + \frac{1}{2} k_2 (B_1 B_2 - A_1 A_2) \sin(k_1 + k_2) x \\
 &+ \frac{1}{2} k_2 (A_1 B_2 - A_2 B_1) \cos(k_1 - k_2) x + \frac{1}{2} k_2 (B_1 B_2 + A_1 A_2) \sin(k_1 - k_2) x \quad (13.1.2)
\end{align*}
\]

(13.1.2) shows that the effect of the non-linear term is to create new components with wave numbers \((k_1 + k_2)\) and \((k_1 - k_2)\), where we have assumed that \(k_1 > k_2\). The component with wave number \((k_1 + k_2)\) will have a smaller scale than any of the original components, while the component \((k_1 - k_2)\) will have a scale larger than \(k_1\), because \(k_1 - k_2 < k_1\). The scale of \((k_1 - k_2)\) may even be larger than \(k_2\), i.e., \(k_1 - k_2 < k_2\), provided \(k_1 < 2k_2\). It is thus seen that in principle the non-linear terms change the distribution on the various scales and that the new components, generated by non-linear interaction, may have scales both smaller and larger than the original scales. Such a process is called a cascade-process, and we may talk about a cascade "up the wave numbers", i.e. to the small-scale motion, or "down the wave numbers", i.e. to the large-scale motion.

The purpose of the following sections is to discuss those aspects of atmospheric turbulence which are of special interest to synoptic meteorology. Many aspects of turbulence of great interest to micrometeorologists are left untouched.

### 13.2 Reynolds Stresses

We shall in this section introduce a formal method of separating the motions on large- and small-time scales. The tool used to do this is the time average. Let us consider an arbitrary scalar quantity \( b = b(x,y,z,t) \). We define a running time average by the expression

\[
\bar{b} = \frac{1}{T} \int_{t' = T/2}^{t + T/2} b \, dt \quad (13.2.1)
\]

\( \bar{b} = \bar{b}(x,y,z,t) \) is still a function of time, but smaller scale fluctuations have been averaged out. It is of interest to determine what happens when (13.2.1) is applied
to a time series. Let us express \( b \) as the series

\[
b = B_0 + \sum_n \left( A_n \cos \frac{\nu_n}{2} t + B_n \sin \frac{\nu_n}{2} t \right)
\]  

(13.2.2)

We may substitute (13.2.2) in (13.2.1) and calculate the integral in a straightforward manner. It turns out that \( \bar{b} \) may be written in the following way

\[
\bar{b} = B_0 + \sum_n \left[ R_n \left( A_n \cos \frac{\nu_n}{2} t + B_n \sin \frac{\nu_n}{2} t \right) \right]
\]  

(13.2.3)

where

\[
R_n = \frac{\sin \left( \frac{\nu_n}{2} T \right)}{\frac{\nu_n}{2} T} = \frac{\sin \left( \frac{T}{T_n} \right)}{\frac{T}{T_n}}
\]

(13.2.4)

where we have defined

\[
T_n = \frac{2\pi}{\nu_n}
\]

(13.2.5)

\( R_n \) is called the response function. It is seen from (13.2.4) that \( R_n \to 1 \) when \( T_n \to \infty \), which means that long-period components are essentially untouched by the averaging defined in (13.2.1). We notice otherwise that \( R_n = 0 \) when \( T_n = T \) while \( 0 < R < 1 \) for \( T < T_n < \infty \). The operator (13.2.1) is therefore a smoother because it reduces the amplitude of the components. We find that the components with \( T_n < T \) are all greatly reduced with an infinite number of zeros for \( R_n \), namely all points where \( T_n = T/p \), where \( p = 1, 2, 3, \ldots \). The field \( \bar{b} \) therefore represents mainly long-period motion, while short-period (high frequency) components are averaged out.

The next step is to find the equation for the averaged motion and to find in which way the high-frequency components interact with the low-frequency components. It is convenient for the later mathematical manipulations to define first the average mass transport \( \bar{\rho \nu} \). Having this we define an average velocity \( \bar{\nu} \) by the equation

\[
\bar{\rho} \bar{\nu} = \bar{\rho \nu}
\]

(13.2.6)
or

$$\vec{v} = \frac{\int t + \frac{1}{2} T \rho \vec{v} \, dt}{\int t - \frac{1}{2} T \rho \, dt} \quad (13.2.7)$$

The vector $\vec{v}$ is, according to (13.2.6), the velocity which is needed for mass transport using the averaged density equal to the real averaged mass transport in the atmosphere. We define the fluctuating velocity $\vec{v}'$ by the equation

$$\vec{v} = \vec{v} + \vec{v}' \quad (13.2.8)$$

and we find from (13.2.6), inserting (13.2.8), that

$$\overline{\rho \vec{v}'} = 0 \quad (13.2.9)$$

Using the averaging operator on the continuity equation (2.4.1):

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \quad (13.2.10)$$

we get using (13.2.6)

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho u}}{\partial x} + \frac{\partial \overline{\rho v}}{\partial y} + \frac{\partial \overline{\rho w}}{\partial z} = 0 \quad (13.2.11)$$

Let us next consider the first equation of motion

$$\frac{du}{dt} + u \frac{du}{dx} + v \frac{du}{dy} + w \frac{du}{dz} = - \frac{1}{\rho} \frac{\partial \rho}{\partial x} + f v + e w - \epsilon \frac{\rho}{\rho} x \quad (13.2.12)$$

If we multiply (13.2.10) by $u$ and (13.2.12) by $p$ and add the two resulting equations we get

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u u}{\partial x} + \frac{\partial \rho u v}{\partial y} + \frac{\partial \rho u w}{\partial z} = - \frac{\partial \rho}{\partial x} + f \rho v - e \rho w + p \frac{\partial \rho}{\partial x} \quad (13.2.13)$$
Using an analogous procedure for the second and third equations of motion we get

\[
\frac{dP_v}{dt} + \frac{dp_{uv}}{dx} + \frac{dp_{vv}}{dy} + \frac{dp_{wv}}{dz} = - \frac{dp}{dy} - f \rho u + F_y \tag{13.2.14}
\]

\[
\frac{dp_{w}}{dt} + \frac{dp_{uw}}{dx} + \frac{dp_{vw}}{dy} + \frac{dp_{ww}}{dz} = - \frac{dp}{dz} - g \rho + e \rho + F_z \tag{13.2.15}
\]

We get now:

\[
\overline{p_{uu}} = \overline{p(u+u')} = \overline{pu} \overline{u} + \overline{p(u+u')}u' = \overline{p} \overline{u} \overline{u} + \overline{pu'}u' \tag{13.2.16}
\]

with similar expressions for all the other components. Using (13.2.16) and the analogous expressions we find by averaging (13.2.13) - (13.2.15):

\[
\frac{dp_{w}}{dt} + \frac{dp_{uw}}{dx} + \frac{dp_{vw}}{dy} + \frac{dp_{ww}}{dz} = - \frac{dp}{dx} + f \overline{p} \overline{u} - e \overline{p} \overline{u} \tag{13.2.17}
\]

\[
+ \left( p_x - \frac{dp_{uu}u'}{dx} - \frac{dp_{uw}u'}{dy} - \frac{dp_{ww}u'}{dz} \right) \tag{13.2.17}
\]

\[
\frac{dp_{v}}{dt} + \frac{dp_{uv}}{dx} + \frac{dp_{vv}}{dy} + \frac{dp_{wv}}{dz} = - \frac{dp}{dy} - f \overline{p} \overline{v} \tag{13.2.18}
\]

\[
+ \left( p_y - \frac{dp_{uu}v'}{dx} - \frac{dp_{uv}v'}{dy} - \frac{dp_{ww}v'}{dz} \right) \tag{13.2.18}
\]

\[
\frac{dp_{w}}{dt} + \frac{dp_{uw}}{dx} + \frac{dp_{vw}}{dy} + \frac{dp_{ww}}{dz} = - \frac{dp}{dz} - g \overline{p} + e \overline{p} \overline{w} \tag{13.2.19}
\]

\[
+ \left( p_z - \frac{dp_{uw}w'}{dx} - \frac{dp_{vw}w'}{dy} - \frac{dp_{ww}w'}{dz} \right) \tag{13.2.19}
\]

Equations (13.2.17) - (13.2.19) are the equations for the time-averaged motion. It is very important to note that these equations are completely equivalent to equations (13.2.13) - (13.2.15) for unaveraged motion except for the last three terms in each equation. These terms, the so-called Reynolds stresses, express the
influence of the fluctuating velocity $\mathbf{v}'$ on the development of the field $\mathbf{v}$. It is also seen that these terms act as additional frictional terms. The main problem is to express the Reynolds stresses in terms of the averaged fields, a problem to which we shall return in the following sections.

Let us next consider the thermodynamic equation (2.3.9) which, after use of the continuity equation, becomes

$$
\frac{c_v}{v} \frac{dT}{dt} = -RT \mathbf{v} \cdot \mathbf{v} + H \tag{13.2.20}
$$

or

$$
\frac{dT}{dt} + u \frac{dT}{dx} + v \frac{dT}{dy} + w \frac{dT}{dz} = - \frac{R}{c_v} TV \mathbf{v} \cdot \mathbf{v} + c_v H \tag{13.2.21}
$$

This equation is now combined with the continuity equation (13.2.10) by multiplying (13.2.10) by $T$ and (13.2.21) by $p$ and adding the resulting equations. The result is

$$
\frac{dT^p}{dt} + \frac{d}{dx} (pTu) + \frac{d}{dy} (pTv) + \frac{d}{dz} (pTw) = - \frac{1}{c_v} p \mathbf{v} \cdot \mathbf{v} + \frac{1}{c_v} p H \tag{13.2.22}
$$

which averaged becomes:

$$
\frac{dT^p}{dt} + \frac{d}{dx} (\hat{p}u) + \frac{d}{dy} (\hat{p}v) + \frac{d}{dz} (\hat{p}w) = - \frac{1}{c_v} \mathbf{p} \cdot \mathbf{v} + \frac{1}{c_v} \mathbf{p} \cdot \mathbf{\hat{v}} \tag{13.2.23}
$$

where we have defined for an arbitrary scalar $b$ the average $\hat{b}$ by the relation

$$
\overline{p \hat{b}} = \overline{p b} \tag{13.2.24}
$$
in analogy with (13.2.6). $b'$ is then defined by the relation

$$b = \hat{b} + b'$$  (13.2.25)

We note in particular that the gas equation in averaged form becomes

$$\overline{p} = R \overline{T} \overline{\mu}$$  (13.2.26)

The variables in the equations just developed are $\theta$, $\overline{T}$, $\overline{p}$ and $\overline{\mu}$ plus all the fluctuations appearing in the Reynolds stresses. In order to predict the averaged quantities we must relate the fluctuations to the mean quantities. This problem has not been solved in general. One of the reasons is that the relations between the mean and the fluctuating quantities depend very much on how the time $T$, the averaging time, is defined. If $T$ is relatively short, say 10 min or so we can consider the fluctuations as turbulence, in which case we have reasonably good empirical results for the interactions. In this case ($T$ small), it is found that the Reynolds stresses completely dominate $\overline{\mu}$ which is of a molecular nature. We may thus disregard molecular friction completely in the treatment of atmospheric motion. In a similar way it turns out that the heat flux by the fluctuating quantities is larger than the molecular heat conduction. The latter quantity also therefore need not be included in $H$ in the thermodynamic equation.

Suppose on the other hand that we consider the synoptic scale motion when the stations are a few hundred kilometres apart. The space smoothing conducted in any analysis will filter the motion on a small scale, say less than a few hundred km. We do not know at the present time how to include the interaction between motion on this scale and the synoptic motion represented on weather maps.

13.3 Molecular Viscosity

No fluid or gas is entirely without friction. We may gain some insight into the nature of molecular viscosity by a consideration of the following experiment.

![Figure 13.1](image-url)
Imagine a viscous fluid between two horizontal plates of which the lower plate is kept fixed while the upper plate is moved with a constant velocity \( U_0 \). The fluid next to the upper plate will be dragged along, and it will attain the velocity \( U_0 \), while the fluid at the lower plate will have the velocity zero. At both plates we have then satisfied the kinematic boundary condition for a viscous fluid, i.e. the normal and the tangential velocities relative to the boundary are both zero.

Experience shows that the velocity profile in the interior of the fluid will be linear, i.e. \( U = U(z) = U_0 (z/D) \) where \( D \) is the distance between the plates. It turns out from the experiment that in order to drag the upper plate along with the velocity \( U_0 \) the necessary force (measured per unit area of the plate) is directly proportional to \( U_0 \) and inversely proportional to the distance \( D \). We may write

\[
\tau = \mu \frac{U_0}{D} = \mu \frac{dU}{dz}
\]

(13.3.1)

where \( \tau \) is the stress, i.e. force per unit area, acting in the x-direction.

It is similarly found by a consideration of a thin fluid element, that the fluid above will act on the upper horizontal surface with a certain stress \( \tau (z+\delta z) \delta x \delta y \delta z \), where \( \delta x \), \( \delta y \) and \( \delta z \) are the dimensions of the fluid element. The fluid at the bottom of AB will act on the fluid below with a stress \( \tau(z) \delta x \delta y \). According to the law of action and reaction, the fluid below will act on the bottom of AB with the stress \( -\tau(z) \delta x \delta y \). The net force acting on AB is then

\[
(\tau (z+\delta z) - \tau(z)) \delta x \delta y \delta z = \frac{d\tau}{dz} \delta x \delta y \delta z
\]

(13.3.2)

or, the force per unit mass

\[
\frac{1}{\rho} \frac{d\tau}{dz}
\]

(13.3.3)

The simple experiment described above has only stresses acting in the x-direction because the fluid elements are all moving in that direction. The net force acting on the fluid element is due to the fact that there is velocity shear in the fluid, i.e. \( \frac{dU}{dz} \neq 0 \). In general, we have velocity variations in all three velocity components in all three coordinate directions. It is plausible then that, in addition to the stress considered above and due to velocity shear in the z-direction but with the stress acting in the x-direction, i.e. \( \tau = \tau_{zx} \), we will have stresses \( \tau_{yx} \) and \( \tau_{xx} \) due to the shear in the y-direction and in the x-direction of the velocity \( u = u(x,y,z) \). There will similarly be three contributions to the net force due to
friction \( (\tau_{zy}, \tau_{yy} \text{ and } \tau_{xy}) \) in the y-direction and three contributions \( (\tau_{zz}, \tau_{yz} \text{ and } \tau_{xz}) \) in the z-direction. It is, however, beyond the scope of this compendium to consider the details of all the stresses. We realize from the treatment above that the molecular frictional forces are of the form

\[
P_x = \frac{1}{\rho} \frac{d\tau_{zx}}{dz} + \text{additional components} \tag{13.3.4}\]

and that similar terms appear in the other equations of motion. One of the reasons why we find it unnecessary to treat the molecular frictional stresses in detail is the remark made in the previous section that they are entirely negligible compared to the Reynolds stresses.

The form of \( P_x \) is

\[
P_x = \frac{1}{\rho} \frac{d}{dz} \left( \mu \frac{du}{dz} \right) = \frac{\mu}{\rho} \frac{d^2 u}{dz^2} \tag{13.3.5}\]

where the last expression can be used if \( \mu \) is a constant. \( v = \mu/\rho \) is called the kinematic viscosity coefficient.

13.4 Laminar and Turbulent Flow

The flow considered in the last section is supposed to be a smoothed regular flow. Not all flows are of this kind. This can be seen from the description of some experiments of flow through tubes and pipes. Consider a pipe of diameter \( D \) through which a fluid with kinematic viscosity \( v \) is flowing. The velocity will vary from zero at the walls to a maximum in the centre of the pipe. We may define a characteristic velocity \( U \) by the total volume transport through the cross-section of the diameter divided by the area of the cross-section of the pipe. For a given pipe \( (D = \text{const.}) \) and a given fluid \( (v = \text{const.}) \), one finds that the flow will be laminar with streamlines and trajectories parallel to the diameter of the pipe as long as \( U \) is not too large. However, when \( U \) exceeds a certain value the flow will suddenly change character and become turbulent, i.e. unorganized, irregular, and highly variable in time and space. One can see such a transition from laminar to turbulent flow by introducing a tracer into the fluid. As long as the flow is laminar we will get straight lines from the tracer, while the tracer will show highly irregular patterns when the transition to turbulent flow has taken place.

Reynold who first made experiments of this kind found that the transition from laminar to turbulent flow takes place at a certain critical value of the quantity \( (UD/v) \). We see from (13.3.5) that the dimension of \( v \) is \( [v] = L^2 T^{-1} \) because \( F_x \) has
the dimension \[ [ \frac{F_x}{L} ] = L T^{-2} \] and therefore
\[
[ \nu ] = \frac{L^2 T^{-2}}{LT^{-1} L^{-2}} = L^2 T^{-1}
\]
(13.4.1)

We find therefore that
\[
Re = \frac{\overline{U} D}{\nu}
\]
(13.4.2)

is a non-dimensional number called the Reynold number. The critical Reynold number is approximately 2300 for pipe flow.

The turbulent velocity field will still have an averaged velocity profile across the pipe. We may calculate the velocity using (13.2.1) with a suitable value of \( T \). The laminar flow, described by Poiseuille's law, has a profile given by
\[
u_L = u_L(r) = 2 \overline{u} \left[ 1 - \left( \frac{2r}{D} \right)^2 \right]
\]
(13.4.3)

where \( r \) is the distance from the centre. The profile is therefore a parabola as shown by the curve \( u_L = u_L(r) \) in Figure 13.2.

```
Figure 13.2
```

When turbulence sets in, there will be changes in the wind profile as shown by the curve \( u_T = u_T(r) \) characteristic for turbulent flow.

The formation of turbulence may, to a certain extent, be explained as an instability phenomenon because the pipe flow will become unstable when too large velocity gradients appear at the boundary or in the interior of the fluid. Turbulent flow is therefore the rule rather than the exception. This holds true for atmospheric flow too, especially in the flow next to the ground, but also in the interior as experienced
by an aircraft when it meets so-called clear-air turbulence (CAT) which is found in those regions of the jet-stream where the horizontal and the vertical wind shears are large or when it goes through clouds containing strong convective currents.

A theoretical description of the details of the turbulent flow is, from a meteorological point of view, neither possible nor desirable. We cannot possibly be interested in describing each little turbulent vortex in detail. We are, however, very interested in the influence of the turbulence on the averaged conditions and in the intensity of the turbulence itself. It is thus obvious that the only way open to us is a statistical description of the turbulence in a way similar to the kinetic theory of gases which is also of a statistical nature.

We can expect to find turbulence in regions where the averaged motion (as defined in section 13.2) has large wind shears, i.e. close to the ground or to the jet-streams. In addition, experience shows that turbulence is found in convective systems such as cumulus clouds. It is therefore understandable (see Physical Meteorology, Ch. X) that the maintenance of turbulence depends on the physical processes which are responsible for the initiation and maintenance of the convective currents. It is known that the main factor responsible for convection is the difference between the adiabatic lapse rate and the actual lapse rate. The combined effects of the wind shear and the temperature lapse rate are often given by a consideration of the so-called Richardson number

\[ \text{Ri} = \frac{\frac{\partial \theta}{\partial z}}{\left( \frac{\partial T}{\partial z} \right)^2} \]  

(13.4.4)

where \( \theta \) is the potential temperature. \( \text{Ri} \) is a non-dimensional number since both the numerator and the denominator have the dimension \( T^{-2} (s^{-2}) \).

From the definition of potential temperature and using the hydrostatic equation we find that

\[ \frac{\frac{\partial \theta}{\partial z}}{\frac{\partial \theta}{\partial z}} = \frac{\frac{\partial T}{\partial z}}{T} (\gamma_d - \gamma) \]  

(13.4.5)

It is thus seen that \( \text{Ri} < 0 \) if \( \gamma > \gamma_d \). We therefore require small or negative values of \( \text{Ri} \) in order to maintain convection. On the other hand, we have seen that turbulence sets in when \( |\partial \theta / \partial z| \) is large. Small values of \( \text{Ri} \) are therefore in general favourable for turbulence.
The Richardson number may also be expressed in a form more suitable for the system with pressure as the vertical co-ordinate. We find

$$
Ri = \frac{g}{\rho} \left( - \frac{g}{\rho} \frac{\partial \rho}{\partial p} \right) \frac{\partial g}{\partial p} = - \frac{a}{g} \frac{\partial g}{\partial p} \left( \frac{\partial v}{\partial p} \right)^2 = \frac{\sigma}{(\nabla \cdot \mathbf{v})^2}
$$

(13.4.6)

where we have used the usual notation $\sigma = (- a/g) \times (\partial g/\partial p)$.

The large values of $\nabla \cdot \mathbf{v}/\partial z$ which favour turbulence by making the Richardson number small are found mostly in the layer next to the ground where $Ri$ is very small and $Re$ is very large. Suppose that $\gamma = 0.57 \times 10^{-2}$ deg m$^{-1}$, $T = 300^0$ K and $g \approx 10$ m s$^{-2}$. We find then $(\partial \rho/\partial T)(\gamma_d - \gamma) \approx 10^{-4}$ s$^{-1}$. It is not unusual in this layer to have $|\partial v/\partial z| \approx 1$ m s$^{-1}$ m$^{-1}$. We find therefore $Re \approx 10^{-4}$. The Reynolds number is, on the other hand

$$
Re = \frac{10 \times 10}{10^{-5}} \approx 10^7
$$

(13.4.7)

where we have used $\overline{u} \approx 10$ m s$^{-1}$, $D \approx 10$ m and $\nu \approx 10^{-5}$ m$^2$ s$^{-1}$, all characteristic for the atmosphere close to the ground.

In the free atmosphere, we find in general a value of the Richardson number much larger than the one quoted above because $(\partial \rho/\partial z)^2$ is much smaller here. Typical values of $Ri$ in the free atmosphere are $10^1$ to $10^2$.

13.5 Turbulence in the Boundary Layer

We take our starting point in the Reynolds stresses as given in (13.2.17) - (13.2.19) for momentum and (13.2.23) for temperature. A number of approximations are usually made when these expressions are applied to the atmospheric boundary layer. The first assumption, usually well satisfied, is that the density fluctuations are small compared to the fluctuations in pressure and temperature. Since the thickness of the layer is relatively small, it is usually assumed that the density is a constant. The second assumption is that the vertical variations are much larger than the horizontal variations in (13.2.17) and (13.2.18) while (13.2.19) is replaced by the hydrostatic relation in the usual way. We have then the following terms left in (13.2.17) and (13.2.18), respectively,
\[- \rho \frac{d\bar{w}'u'}{dz} = \frac{\partial \bar{x}}{\partial z}, \quad - \rho \frac{d\bar{w}'v'}{dz} = \frac{\partial \bar{y}}{\partial z} \tag{13.5.1}\]

where the turbulent stresses are

\[\tau_x = -\rho \bar{w}'u', \quad \tau_y = -\rho \bar{w}'v' \tag{13.5.2}\]

Using the same assumptions in (13.2.23), we have the following remaining term: \(-\rho \frac{d(T'w')}{dz}\) which is proportional to the convergence of the turbulent vertical heat flux. We define

\[H_d = + \frac{c_p}{\rho} \rho \bar{T}' \bar{w}' \tag{13.5.3}\]

as the turbulent vertical transport of sensible heat, and the term appearing in (13.2.23) may be written

\[+ \frac{l}{c_p} \frac{dH_d}{dz} \tag{13.5.4}\]

We are also interested in the vertical transport of water vapour

\[E = \rho m' \bar{w}' \tag{13.5.5}\]

where \(m\) is the specific humidity, because the transport of latent heat is

\[H_L = L \rho m' \bar{w}' = L E \tag{13.5.6}\]

where \(L\) is the latent heat coefficient. \(E\) is equal to the evaporation when it is measured close to the ground.

The requirement, as mentioned before, is to relate the vertical turbulent transports of momentum, heat and moisture, i.e. \(\bar{u}'w', \bar{v}'w', \bar{T}'w'\) and \(\bar{m}'w'\), to the averaged conditions. The classical way of doing this is by applying the so-called mixing length hypothesis, which is explained below.
Consider a quantity $b$ defined per unit mass of air (such as a concentration, the relative humidity etc.), but let it be assumed that $b$ is a conservative quantity, i.e. $\frac{db}{dt} = 0$. Since we are dealing with vertical variations we shall assume that $b$'s field distribution is given by $\overline{b} = \overline{b}(z)$, shown in Figure 13.3, where it has been assumed that $\frac{\partial \overline{b}}{\partial z} < 0$. Consider now a particle which arrives at level $z$ from the level $z - \delta z$. This particle will have a positive vertical velocity $w' > 0$. It carries a value of $\overline{b}$ equal to $\overline{b}(z - \delta z)$. When the particle is at $z = z$ we have a value:

$$b' = \overline{b}(z - \delta z) - \overline{b}(z) = - \frac{\partial \overline{b}}{\partial z} \delta z \quad (13.5.7)$$

where $b' > 0$ (see Figure 13.3 and (13.5.7) when $\frac{\partial \overline{b}}{\partial z} < 0$). We thus find a positive correlation between $b'$ and $w'$. This can also be seen if we consider a particle arriving at $z$ from the level $z + \delta z$. Such a particle would have $w' < 0$, and due to the conservation of $b$ it would have:

$$b' = \overline{b}(z + \delta z) - \overline{b}(z) = \frac{\partial \overline{b}}{\partial z} \delta z \quad (13.5.8)$$

where it is seen that $b' < 0$. We may thus consider (13.5.7) as general as long as we use the convention that $\delta z < 0$ if $w' < 0$. For the turbulent vertical flux of $b$ we find

$$F_b = \overline{p b' w'} = - \overline{\rho (\delta z) w'} \frac{\partial \overline{b}}{\partial z} \quad (13.5.9)$$

Now, the correlation $\overline{\delta z w'}$ is definitely positive according to the convention stated above. We may thus define

$$A = \overline{\rho (\delta z) w'} > 0 \quad (13.5.10)$$
where \( A \) is called the exchange coefficient. We may therefore write

\[
F_b = - A \frac{\partial b}{\partial z} \tag{13.5.11}
\]

Since \( A > 0 \), \( \partial b/\partial z < 0 \), we find \( F_b > 0 \). According to the formulation given here we always have a flux directed from large to small values of \( b \) (in Figure 13.3: upwards). It is customary to introduce still another coefficient \( K = A/\rho \) such that

\[
F_b = - \rho K \frac{\partial b}{\partial z} \tag{13.5.12}
\]

where \( K \) is called the eddy diffusivity.

The coefficients \( A \) and \( K \) are generally not constants because they must depend upon the intensity of the turbulence according to (13.5.10). Large values of \( w' \) give large values of \( \partial z \) and thus large values of \( A \) (and \( K \)). On the other hand, the quantities favouring turbulence are large values of \( \partial \theta/\partial z \) and small values of \( g \frac{d\ln \theta}{dz} \), i.e. small values of \( Ri \). We must therefore expect that \( K \) will be a function of these quantities.

We shall next apply the mixing length theory to some cases. For the case of the potential temperature we find \( b = \theta \) and \( d\theta/dt = 0 \) (adiabatic displacement). We can use this to express the turbulent heat flux \( H_\theta \). From the thermodynamic equation we have

\[
c_p \frac{dT}{dt} - a \frac{dp}{dt} = 0 \tag{13.5.13}
\]

but

\[
\omega = \frac{dp}{dt} \approx - gpw \tag{13.5.14}
\]

and

\[
a \frac{dp}{dt} = gw = + g \frac{dz}{dt} \tag{13.5.15}
\]

Inserting into (13.5.13) we get
\[
\frac{d}{dt} \left[ T + \gamma_d z \right] = 0
\]  
(13.5.16)

We may therefore just as well set \( q = T + \gamma_d z \) and

\[
\frac{H_d}{c_p} = \rho \overline{u'w'} = - \rho \, K_H \left( \frac{dT}{dz} + \gamma_d \right)
\]  
(13.5.17)

or, finally

\[
H_d = - c_p \, K_H \rho \left( \gamma_d - \overline{\gamma} \right)
\]  
(13.5.18)

When it comes to the vertical transport of momentum, we do not have a conservative quantity analogous to the potential temperature. The justification for using (13.5.12) is therefore not as valid as before, but it is usually assumed that \( u \) is conserved in these relatively small vertical displacements. We get then (see (13.5.2))

\[
- \tau_x = \rho \overline{u'u''} = - \rho \, K_M \frac{d\overline{u}}{dz}
\]  
(13.5.19)

\[
- \tau_y = \rho \overline{v'w'} = - \rho \, K_M \frac{d\overline{v}}{dz}
\]

but there is no guarantee that \( K_M = K_H \).

13.6 The Prandtl-layer

The lowest layer next to the ground is called the Prandtl-layer. It is characterized by rapid variations of \( \overline{u}, \overline{T} \) and \( \overline{w} \) with regard to height. We must also expect that \( K \) becomes very small next to the ground because \( w \) becomes small.

The Prandtl-layer is quite thin (20-60 m) and has therefore little mass per unit area. It is defined as the layer through which one may assume that \( \tau, H_d \), and \( H_e \) are constants with respect to height. Considering only the x-direction, we have

\[
\tau_x = - \rho \overline{u'u''} = \rho \overline{u_x'^2}
\]  
(13.6.1)

where we have assumed that \( \overline{u'u''} < 0 \) (downward transport) and where (13.6.1) should be considered as the defining equation for \( u_x \), called the friction velocity.
Since $\tau = \text{const}$, $\rho = \text{const}$, we have $u_* = \text{const}$ in the Prandtl-layer. $u_*$ is therefore just another way of expressing the stress $\tau$, i.e.

$$u_* = \sqrt{\frac{\tau}{\rho}} \tag{13.6.2}$$

It is equally convenient to introduce a friction temperature $T_*$ and a friction specific humidity $m_*$ by the defining equations

$$-\frac{H_d}{c_p} = -\rho \frac{\overline{w'}}{w'} = \rho u_* T_* \tag{13.6.3}$$

$$-E = -\rho \frac{m'}{m'} = \rho u_* m_* \tag{13.6.4}$$

$T_*$ and $m_*$ are constants in the Prandtl-layer, and $T_* > 0$ if the flux of sensible heat is downwards, i.e. $\overline{w'} < 0$. In a similar way we find that $m_* > 0$ if $\overline{m'} < 0$.

We have now the following equations

$$K_H \frac{\overline{u}}{\overline{u}_*} = u_*^2 \quad \text{(from (13.5.9))} \tag{13.6.5}$$

$$K_H \left(\frac{\overline{\overline{u}}}{\overline{u}_*} + \gamma_d\right) = u_* T_* \quad \text{(from (13.5.17))} \tag{13.6.6}$$

$$K \frac{\overline{m}}{\overline{m}_*} = u_* m_* \tag{13.6.7}$$

where the last equation is obtained by using (13.5.12) with $b = m$. These equations can be used to calculate $\overline{u}$, $\overline{\overline{u}}$, and $\overline{m}$ provided we know $K_H$, $K_H$, and $K$ as functions of $z$, the distance from the ground.

(a) Neutral Stratification

We are interested in one case here, i.e. the case where $H_d = 0$. We have then from (13.6.3) that $T_* = 0$ and therefore

$$\frac{\overline{\overline{u}}}{\overline{u}_*} + \frac{\gamma_d}{c_p} = 0 \tag{13.6.8}$$
i.e. neutral stratification \( \bar{\nu} = \nu \). Since we decided earlier that the molecular processes are, in general, negligible, we can say that the only parameters which can determine \( K_M \) are \( \frac{jU}{jz} \) and \( z \). However, in view of (13.6.5) we may also say that \( K_M \) depends upon \( u^* \) and \( z \). Now \( K_M \) has the dimension \( L^2 T^{-1} \) while \( [u^*] = L T^{-1} \) and \( [z] = L \).

It follows therefore that

\[
K_M = \nu u^* z
\]  

(13.6.9)

where \( \nu \) is von Karmans constant. Empirically, it has been found that \( \nu = 0.4 \).

Substituting (13.6.9) in (13.6.5) we find

\[
\nu u^* \frac{jU}{jz} = u^2
\]  

(13.6.10)

or

\[
\int_0^{\bar{u}} \frac{jU}{u^*} = \frac{1}{\nu} \int_0^{\bar{z}} \frac{jz}{z}
\]  

(13.6.11)

which integrates to

\[
\bar{u} = \frac{u^*}{\nu} \ln\left(\frac{z}{z_0}\right)
\]  

(13.6.12)

where \( z_0 \), the roughness length, is the integration constant defined as the place where \( \bar{u}(z) = u(z_0) = 0 \). (13.6.12) is the logarithmic wind profile, first derived by Prandtl. It is well verified by data, as long as the condition (13.6.3) is reasonably fulfilled, i.e. \( H_d \) is small.

(13.6.12) can apparently not be used all the way to \( z = 0 \). The vegetation or the nature of the soil is important over land, while the waves are the main feature of importance over the ocean, determining that \( K_M \) does not quite go to zero at \( z = 0 \) but rather at a height \( z = z_0 \) which is related to the nature of the terrain. For a plane surface covered by sand one gets from experimentation that \( z_0 \approx 10^{-2} \) to \( 10^{-1} \) cm, while \( z_0 \) over grass varies from about 0.5 cm to 10 cm depending on the length of the grass. For the ocean, we find that \( z_0 \) is a function of the effective wave height and therefore of the wind speed, but \( z_0 \) is normally considerably less than 1 cm.
In meteorological literature we often find (13.6.12) replaced by
\[ \bar{u} = \frac{u_*}{x} \ln \left( 1 + \frac{z}{z_o} \right) \]  
(13.6.13)

which is obtained by replacing \( z \) by \( z + z_o \). The only advantage of (13.6.13) is the fact that \( \bar{u}(0) = 0 \), but otherwise it is of course only a change of the zero point for the co-ordinate system.

We can use (13.6.12) or (13.6.13) to determine the friction velocity \( u_* \) and the roughness height \( z_o \) and thereby the stress \( \tau \) from measurements of \( \bar{u}(z) \). The measured values of \( \bar{u}(z) \) are plotted on a co-ordinate system which is linear on the ordinate and logarithmic on the abscissa, i.e. \( \xi = \ln z, \xi_o = \ln z_o \). If the measured profile should satisfy (13.6.12), the points \( \bar{u}(z) \) will be on a straight line in the diagram, i.e.
\[ \bar{u}(z) = \frac{u_*}{x} \left( \xi - \xi_o \right) \]  
(13.6.14)

\( u_*/x \) is the slope of the line. Since \( x = 0.4 \), we have \( u_* \), and \( \tau \) can be computed from \( \tau = \rho u_*^2 \). Similarly, the intersection of the line with the abscissa gives the point where \( \bar{u} = 0 \), i.e. where \( \xi = \xi_o = \ln z_o \). \( z_o \) can therefore be computed.

We recall that (13.6.12) was obtained on the assumption that the molecular viscosity was of no importance in the problem. While in general this assumption is applicable, there are some exceptions. The details of these empirical investigations will not be given here, but it turns out that the assumption is well justified if the Reynolds number
\[ Re = \frac{z_o u_*}{v} > 2.5 \]  
(13.6.15)

On the other hand, if the roughness and/or the stress are so small that
\[ Re = \frac{z_o u_*}{v} < 0.13 \]  
(13.6.16)

we are not permitted to neglect the quantity \( v \), i.e. the molecular viscosity.

In the first case (\( Re > 2.5 \)), we call the surface aerodynamically rough,
while it is termed \textit{aerodynamically smooth} if \( \text{Re} < 0.13 \). In the latter case, we have a thin layer next to the ground where molecular viscosity is dominating, i.e. the so-called laminar sub-layer, in which

\[
\frac{\tau}{\rho} = u_x^2 = v \frac{d\bar{u}}{dz} \tag{13.6.17}
\]

from which

\[
\bar{u} = v^{-1} u_x^2 z \tag{13.6.18}
\]

We thus have a linear wind profile in the laminar sub-layer. It should be stressed, however, that (13.6.15) is normally satisfied in the atmospheric boundary layer, and hence we may call the earth surface aerodynamically rough.

The next part of this section will be devoted to a useful determination of the surface stress. Combining (13.6.12) with (13.6.1) by eliminating \( u_x \) we find

\[
\tau = \rho \frac{\kappa}{\left( \ln \left( \frac{z}{z_0} \right) \right)^2} u^2 \tag{13.6.19}
\]

or

\[
\tau = c_d \bar{u}^2 \tag{13.6.20}
\]

where

\[
c_d = \frac{\kappa}{\left( \ln \left( \frac{z}{z_0} \right) \right)^2} = \frac{u_x^2}{\bar{u}^2} \tag{13.6.21}
\]

\( c_d \), the drag coefficient defined in (13.6.21) is a non-dimensional number, but is naturally not a constant, but a function of \( z \). (13.6.20) is very useful in expressing \( \tau \) in terms of wind velocity squared. If we use the value of \( \bar{u} \) at \( z = 10 \text{ m} \), we must naturally calculate \( c_d \) from (13.6.21) using the same value of \( z \).

If we measure the wind at two levels, and if we know that the logarithmic wind law applies, we find from (13.6.12)
\[ \bar{u}_2 - \bar{u}_1 = \frac{u_*}{\kappa} \ln \left( \frac{z_2}{z_1} \right) \] (13.6.22)

an equation which does not contain \( z_0 \). It may be used to calculate the friction velocity from measurements of the wind at two heights in the Prandtl-layer.

With the information about \( \bar{u} = \bar{u}(z) \) we may now turn to the vertical turbulent transport of moisture, i.e. to (13.6.7). Substituting from (13.6.9) we find

\[ k z \frac{d \bar{m}}{dz} = \bar{m}_* \] (13.6.23)

Note that we have assumed \( K = K_H \). Integration of (13.6.23) gives

\[ \bar{m} = \frac{\bar{m}_*}{\kappa} \ln \left( \frac{z}{z_{\text{om}}} \right) \] (13.6.24)

where \( z_{\text{om}} \) is the level at which \( \bar{m} = 0 \). \( z_{\text{om}} \) is therefore analogous to the roughness length, but it is not the same quantity and will be in general numerically different, because moisture is transferred from the ground to the air by molecular processes, while momentum may be transferred by pressure forces. Using two-level measurements in the logarithmic layer for \( \bar{m} \) we find from (13.6.24) that

\[ \bar{m}_2 - \bar{m}_1 = \frac{\bar{m}_*}{\kappa} \ln \left( \frac{z_2}{z_1} \right) \] (13.6.25)

having eliminated \( z_{\text{om}} \).

We may now express the expression \( E \) in (14.6.4) in the following way

\[ E = - \rho u_* m_* = - \rho \frac{\kappa}{\ln \left( \frac{z_2}{z_1} \right)} (\bar{m}_2 - \bar{m}_1) u_* \] (13.6.26)

where we have substituted from (13.6.25). From (13.6.22) we may solve for \( \kappa / \ln(z_2/z_1) \) and substitute in (13.6.26). We get

\[ E = - \rho \frac{\bar{m}_2 - \bar{m}_1}{u_2 - \bar{u}_1} u_*^2 \] (13.6.27)
which, using (13.6.21), may be written

\[ E = - \rho \frac{\bar{m}_2 - \bar{m}_1}{\bar{u}_2 - \bar{u}_1} c_d \bar{u}^2 \]  \hspace{1cm} (13.6.28)

We may write (13.6.23) in a more convenient way by defining that level 1 should be the surface giving \( \bar{u}_1 = 0 \) according to the boundary condition, and letting level 2 coincide with the level at which \( \bar{u} \) (and \( c_d \)) are measured, i.e. \( \bar{u}_2 = \bar{u} \).

Setting \( \bar{m}_2 = m \) and \( \bar{m}_1 = m_S \) (\( s = \) surface)

\[ E = - \rho c_d (\bar{m} - \bar{m}_S) \bar{u} \]  \hspace{1cm} (13.6.29)

a formula which may be used to calculate the evaporation from measurements of \( c_d \), \( \bar{u} \), and \( \bar{m} \) at the same level in the logarithmic layer and from a measurement of \( \bar{m}_S \).

A word of caution should be added here: (13.6.23) to (13.6.29) were all derived by assuming that \( K \) (for the moisture) is equal to \( K_M \) (for the momentum). This is not always true. When (13.6.29) is used in practice it is common to replace \( c_d \) by \( c \), called the evaporation coefficient, and to determine \( c \) empirically. Finally, note that \( \bar{m}_S \) in (13.6.29) can be obtained from \( \bar{m}_S \), the measured temperature at the sea surface, assuming saturation at that surface.

(b) **Stable and Unstable Stratification**

The detailed treatment given above applies for neutral stratification only. It is much more difficult to give an equally detailed treatment for stable and unstable cases, although one such theory has been advanced by Monin and Obukhov. **This theory, however, is beyond the scope of the present treatment.** We shall be satisfied with an empirical description of the various stratifications developed by Deacon, who has investigated numerous vertical wind profiles under a wide range of stratifications. He concludes that these profiles can be described by setting

\[ \frac{\partial \bar{u}}{\partial z} = \frac{u_*}{k z_o} \left( \frac{z}{z_o} \right) - \beta \]  \hspace{1cm} (13.6.30)

where the symbols have the same meaning as before and where \( \beta \) is determined empirically. Deacon used a Richardson number as a measure of the stratification and found that \( \beta \) is a decreasing function of the Richardson number such that \( \beta = 1 \) when \( Ri = 0 \). (13.6.30) is therefore a generalization of the logarithmic wind profile as can be seen by setting \( \beta = 1 \). He found \( \beta > 1 \), when \( Ri < 0 \) (unstable), and \( \beta < 1 \), when \( Ri > 0 \) (stable).
(13.6.30) may be integrated to give
\[
\bar{u} = \frac{u^*}{x(1-\beta)} \left( \frac{z}{z_0}^{1-\beta} - 1 \right) \tag{13.6.31}
\]

A major empirical result of Deacon's investigation is that (13.6.30) and (13.6.31) apply for a wide range of Richardson numbers. It follows then that \( z_0 \) is independent of the stratification. The roughness length is therefore a number characterizing the surface, and \( z_0 \) may be determined in cases of neutral stratification for which the logarithmic wind law applies.

(13.6.31) is well suited to describe the diurnal variation of the wind profile in the lowest layer of the atmosphere. During the day-time, we will have values of the lapse rate close to or even exceeding \( \gamma_d \). \( R_i \) will therefore be small or negative, leading to values of \( \beta > 1 \). On the other hand, during the night with radiational cooling, we get a very stable stratification, \( R_i \) becomes large and \( \beta < 1 \). The schematic wind profiles are shown in Figure 13.4 where the ordinate is \( \ln z \) such that the logarithmic wind profile becomes a straight line.

![Figure 13.4](image)

13.7 The Planetary Boundary Layer

The direct influence of surface friction on atmospheric flow is seen up to a height \( H \), which may be called the level of direct frictional influence. \( H \) may be defined as the level at which the wind approaches the geostrophic (or gradient) wind. The layer above \( z = H \) is called the "free" atmosphere, while the layer below \( z = H \) is called the planetary boundary layer. At the very bottom of this layer we have a laminar sub-layer, but as was pointed out earlier, this exists only on rare occasions. Otherwise the lowest layer is the Prandtl layer treated in detail in section 13.6. The depth \( h \) of the Prandtl layer is of the magnitude
20-50 m, and it is characterized by a constant value of the stress in the vertical direction, and, furthermore, the stress is much larger than the pressure force and the Coriolis force in the Prandtl layer. From the results obtained in section 13.6 we note in particular that, in the Prandtl layer, the wind vectors vary in magnitude but not in direction. This may also be expressed in the form that the thermal wind, $\frac{\partial \mathbf{v}}{\partial z}$, is parallel to the wind itself in this layer.

As we move higher into the atmosphere we will gradually approach regions where the stress is of smaller, but not negligible, magnitudes. We have then a layer in which the pressure force, the Coriolis force, and the frictional force, expressed in terms of Reynolds stresses, are of equal orders of magnitude. The wind variation in this layer is quite different from the Prandtl layer, and we shall in the next section obtain a simplified theory for the variation. The layer is called the Ekman-layer due to the fact that solutions of the type were first obtained by Ekman although for ocean areas. Because of the geometrical shape of the hodograph in the Ekman-layer it is also called the spiral layer.

13.5 The Ekman-layer

The basic assumption made in the Ekman-layer is a balance of the pressure force, the Coriolis force and the frictional force, or

$$0 = -\frac{1}{\rho} \nabla p - f \mathbf{k} \times \mathbf{v} + \frac{1}{\rho} \frac{\partial \mathbf{v}}{\partial z}$$  \hspace{1cm} (13.3.1)

where we have considered the horizontal forces. It is convenient to express the pressure force in terms of the geostrophic wind, i.e.
\[-\frac{1}{\rho} \nabla p = f \vec{k} \times \vec{v} \]  \hspace{1cm} (13.8.2)

and (13.8.1) becomes
\[f \vec{k} \times (\vec{v} - \vec{v}_g) = \frac{1}{\rho} \frac{d\vec{v}}{dz} \] \hspace{1cm} (13.8.3)

We may illustrate the balance of forces from (13.8.3). In Figure 13.6 we start from the positions of the horizontal wind \(\vec{v}\) and the geostrophic wind \(\vec{v}_g\). We construct the vector \(\vec{v} - \vec{v}_g\), the ageostrophic wind component. It is then known

![Figure 13.6](image)

that the vector \(f \vec{k}\) \((\vec{v} - \vec{v}_g)\) is perpendicular to \(\vec{v} - \vec{v}_g\) and, according to (13.8.3), equal to \(\rho^{-1} \frac{d\vec{v}}{dz}\). In addition, we know that the pressure force is perpendicular to \(\vec{v}_g\), while the Coriolis force is perpendicular to \(\vec{v}\) with the directions indicated on the figure. (13.8.1) now expresses a balance of the three forces shown by double-lined vectors on Figure 13.6. The problem is now to find \(\vec{v}\) when we have expressed \(\rho^{-1} \frac{d\vec{v}}{dz}\) in terms of the wind field. We use (13.5.19) and, assuming that \(\rho \vec{k}\) is independent of \(z\), find that
\[\frac{1}{\rho} \frac{d\vec{v}}{dz} = K \frac{d^2\vec{v}}{dz^2} \] \hspace{1cm} (13.8.4)

and thereby
\[K \frac{d^2\vec{v}}{dz^2} - f \vec{k} \times \vec{v} + f \vec{k} \times \vec{v}_g = 0 \] \hspace{1cm} (13.8.5)

which is the equation which must be solved. It is convenient to introduce a
co-ordinate system with the x-axis along the geostrophic wind, i.e. \( \vec{V}_g = V_g \vec{e}_x \). We have then

\[
K \frac{\partial^2 u}{\partial z^2} + fv = 0
\]

\[
K \frac{\partial^2 v}{\partial z^2} - f(u - V_g) = 0
\]

In the solution of the system (13.8.6) we shall now make the assumption that \( V_g = \text{const.} \). This is equivalent to saying that the pressure force has no variation with height or that the geostrophic thermal wind vanishes. It is seen clearly from (13.8.6) that we are dealing with a set of coupled differential equations. For the purpose of solution it is most convenient to use complex numbers. We define:

\[
W = (u - V_g) + iv
\]

The second equation in (13.8.6) is multiplied by the imaginary number \( i = \sqrt{-1} \) and added to the first equation to give

\[
K \frac{\partial^2 W}{\partial z^2} - ifW = 0
\]

because \( \frac{\partial^2 V_g}{\partial z^2} = 0 \) because \( V_g = \text{const.} \), and because

\[
fv - if(u - V_g) = - f \left[(u - V_g) + iv\right] = -ifW
\]

We write (13.8.9) in the form

\[
\frac{\partial^2 W}{\partial z^2} - a^2 W = 0
\]

where

\[
a^2 = \frac{if}{K}
\]

The solution to (13.8.10) is

\[
W = C_1 e^{az} + C_2 e^{-az}
\]
where \( C_1 \) and \( C_2 \) are constants of integration to be determined from the boundary conditions. We have now

\[
a = \sqrt{\frac{f}{K}} \sqrt{1 + \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)} = \sqrt{\frac{f}{2K}} (1+i) = \frac{1+i}{H_*} \quad (13.8.12)
\]

where

\[
H_* = \sqrt{\frac{2K}{f}}
\]

and \( H_* \) has the dimension of a length because \([K] = L^2 T^{-1}\) and \([f] = T^{-1}\). (13.8.13) may then be written

\[
W = C_1 e^{\frac{(1+i)z}{H_*}} + C_2 e^{-(1+i)\frac{z}{H_*}} \quad (13.8.14)
\]

We turn next to the boundary conditions. At great heights \((z \to \infty)\) we want the wind to approach the geostrophic wind, i.e. \(W \to 0\) as seen from (13.8.7). It follows then that \(C_1 = 0\). Letting \(C_2 = W_0\) we have then from (13.8.14)

\[
W = W_0 e^{-(1+i)\frac{z}{H_*}} \quad (13.8.15)
\]

where \(W_0\) is equal to \(W\) at \(z = 0\) as seen from (13.8.15). \(W_0\) is the remaining integration constant, and because \(W_0\) is a complex constant, we need two conditions to determine it. One of these conditions, mentioned in section 13.7, is that the thermal wind is parallel to the wind itself in the Prandtl-layer. We shall use this condition at the bottom of the Ekman-layer where, for convenience, we put \(z = 0\). The first condition is then

\[
\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} \quad (13.8.16)
\]

The second condition must be that the magnitude of the stress in the Prandtl-layer is equal to the stress at the bottom of the Ekman-layer, which may be written (see (13.8.19))

\[
\tau_0 = \rho_o K \left( \frac{\partial v}{\partial z} \right)_{z = 0} \quad (13.8.17)
\]
However, for the magnitude of $\tau_o$ we have only the formula (13.6.20), and
the second boundary condition would be

$$
\rho_o \left| \frac{\partial v}{\partial z} \right|_{z=0} = c_d \rho_o \omega_o^2
$$

(13.8.19)

(13.8.19) is much less satisfactory than (13.3.16) because the formula for
the magnitude of the stress in the Prandtl-layer is true only under neutral stratifi-
cation. It is therefore customary to disregard (13.8.19) and use (13.3.16) leaving
only one undetermined quantity, which normally is taken to be the angle $\alpha_o$ between
the wind at $z=0$ and the geostrophic wind.

We shall now prepare for the use of (13.3.16). Inserting from (13.9.7)
into (13.8.15), we get

$$
W = (u-V_g) + iv = \left[ (u_o - V_g) + i v_o \right] \cdot \left[ \cos \frac{z}{H_*} - i \sin \frac{z}{H_*} \right] e^{-\frac{z}{H_*}}
$$

(13.8.19)
or, equating real and imaginary parts:

$$
u - V_g = e^{-\frac{z}{H_*}} \left[ (u_o - V_g) \cos \frac{z}{H_*} + v_o \sin \frac{z}{H_*} \right]
$$

(13.8.20)

$$
v = e^{-\frac{z}{H_*}} \left[ v_o \cos \frac{z}{H_*} - (u_o - V_g) \sin \frac{z}{H_*} \right]
$$

We need $\partial u/\partial z$ and $\partial v/\partial z$ and obtain from (13.8.20)

$$
\frac{\partial u}{\partial z} = \frac{1}{H_*} e^{-\frac{z}{H_*}} \left[ - (u_o - V_g) \left( \sin \frac{z}{H_*} + \cos \frac{z}{H_*} \right) + v_o \left( \cos \frac{z}{H_*} - \sin \frac{z}{H_*} \right) \right]
$$

(13.8.21)

$$
\frac{\partial v}{\partial z} = \frac{1}{H_*} e^{-\frac{z}{H_*}} \left[ - v_o \left( \sin \frac{z}{H_*} + \cos \frac{z}{H_*} \right) - (u_o - V_g) \left( \cos \frac{z}{H_*} - \sin \frac{z}{H_*} \right) \right]
$$

Using (13.8.20) and (13.8.21) with $z=0$, condition (13.8.10) becomes

$$
\frac{1}{H_*} \left[ v_o - (u_o - V_g) \right] = \frac{1}{H_*} \left[ v_o + (u_o - V_g) \right]
$$

(13.8.22)
from which we get

\[(u_o - v_o) V_g = u_o^2 + v_o^2 \quad (13.8.23)\]

It is at this point that we introduce the magnitude \( V_o \) and the direction \( a_o \) of the wind at \( z = 0 \). We have \( u_o = V_o \cos a_o, v_o = V_o \sin a_o \), and inserting into \( (13.8.23) \), we obtain

\[V_o = V_g (\cos a_o - \sin a_o) \quad (13.8.24)\]

Using \( (13.8.24) \) we find that

\[u_o - V_g = V_g (\cos a_o - \sin a_o) \cos a_o - V_g = - V_g (\sin a_o + \cos a_o) \sin a_o \]

\[v_o = V_g (\cos a_o - \sin a_o) \sin a_o = - V_g (\sin a_o - \cos a_o) \sin a_o \quad (13.8.25)\]

These expressions may be written in a simpler form by noting that

\[\sin a_o + \cos a_o = \sqrt{2} \left( \sin a_o \sin \frac{3\pi}{4} - \cos a_o \cos \frac{3\pi}{4} \right) = - \sqrt{2} \cos (a_o + \frac{3\pi}{4}) \quad (13.8.26)\]

\[\sin a_o - \cos a_o = \sqrt{2} \left( \sin a_o \cos \frac{3\pi}{4} - \cos a_o \sin \frac{3\pi}{4} \right) = - \sqrt{2} \sin (a_o + \frac{3\pi}{4}) \]

and

\[u_o - V_g = \sqrt{2} V_g \sin a_o \left( \cos a_o + \frac{3\pi}{4} \right) \quad (13.8.27)\]

\[v_o = \sqrt{2} V_g \sin a_o \left( \sin a_o + \frac{3\pi}{4} \right)\]

After all these preparations we may finally write the solution in the form

\[u = V_g + \sqrt{2} V_g \sin a_o \cos \left( a_o + \frac{3\pi}{4} - \frac{z}{H_*} \right) \quad (13.8.28)\]

\[v = - \sqrt{2} V_g \sin a_o \sin \left( a_o + \frac{3\pi}{4} - \frac{z}{H_*} \right)\]

by substitution of \( (13.8.27) \) into \( (13.8.20) \).
It is convenient for discussion of the result to introduce the notation

\[ \mathbf{v}_a = (u - \mathbf{v}_g) \mathbf{i} + \mathbf{v}_f \]  \hspace{1cm} (13.8.29)

and call \( \mathbf{v}_a \) the ageostrophic wind

\[ \mathbf{v}_a = \sqrt{2} \mathbf{v}_g \sin \alpha_0 e^{-\frac{z}{H_*}} \]  \hspace{1cm} (13.8.30)

According to this expression we may characterize the height \( H_* \) as the level where the ageostrophic wind has decreased by a factor \( e = 2.7 \). The actual depth assigned to the Ekman-layer is somewhat arbitrary, but it is normally taken as the depth \( nH_* \). That such a choice is reasonable can be seen from the following table which shows that the ageostrophic wind at \( z = nH_* \) is only 4 percent of the magnitude of the ageostrophic wind at \( z = 0 \) which is \( \sqrt{2} \mathbf{v}_g \sin \alpha_0 \).

<table>
<thead>
<tr>
<th>( \frac{z}{H_*} )</th>
<th>0</th>
<th>( \frac{n}{4} )</th>
<th>1</th>
<th>( \frac{n}{2} )</th>
<th>( \frac{3n}{4} )</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{v}_a )</td>
<td>1</td>
<td>0.46</td>
<td>0.37</td>
<td>0.21</td>
<td>0.10</td>
<td>0.04</td>
</tr>
<tr>
<td>( \sqrt{2} \mathbf{v}_g \sin \alpha_0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 13.8 shows the Ekman spiral calculated for a value of $q_0 = 20^\circ$. The values of $z/H_*$ are marked at the various points on the curve and this verifies that the effective depth is $nH_*$ or

$$D_{EK} = nH_* = \pi \sqrt{\frac{2K}{f_0}}$$  \hspace{1cm} (13.8.31)

A reasonable value of $K$ is $5 \text{ m}^2 \text{ s}^{-1}$ Taking $f_0 = 10^{-4} \text{ s}^{-1}$, we find $D_E \approx 10^3 \text{ m} = 1 \text{ km}$, which may then be taken as the order of magnitude of the thickness of the layer in middle latitudes.

It can be seen from Figure 13.8 that the wind in the Ekman-layer blows across the isobars from higher to lower pressure. There is thus a systematic mass transport across the isobars. We may find the mass transport by a consideration of (13.8.3) giving

$$\frac{\partial \vec{\tau}}{\partial z} = f\vec{k} \times (\rho \vec{v} - \rho_0 \vec{v}_g)$$  \hspace{1cm} (13.3.32)

or

$$\rho \vec{v} = \rho_0 \vec{v}_g - \frac{1}{f} \vec{k} \times \frac{\partial \vec{\tau}}{\partial z}$$  \hspace{1cm} (13.8.33)

The total mass transport in the Ekman-layer is

$$\int_0^{D_E} \rho \vec{v} \, dz = \rho_0 \vec{v}_g \, D_E + \frac{1}{f} \vec{k} \times \vec{T}_o$$  \hspace{1cm} (13.8.34)

where we have used the fact that the stress is vanishingly small at $z = D_E$. 
(13.8.34) shows that the mass transport may be considered as consisting of two parts: a geostrophic transport along the isobars and an ageostrophic transport directed perpendicularly by the surface wind because \( \vec{\tau}_0 \) has the same direction as \( \vec{v}_o \) in the Prandtl-layer. The consequence of the ageostrophic part of the mass transport in the Ekman-layer is that there must be systematic vertical velocities induced by the surface stress. Imagine for example a closed isobar around a low pressure centre. Because of the cross-isobaric flow there will be a tendency to accumulate mass inside the isobar. However, because of the mass continuity equation the convergence of the mass flux will result in positive vertical velocities above the low pressure area. Similarly, there will be mass divergence across a closed isobar surrounding a high pressure centre resulting in sinking motion in the area. We shall now put these considerations into a quantitative measure of the induced vertical velocity.

If we disregard the variation of the Coriolis parameter we find \( \nabla \cdot \vec{v}_e = 0 \). We get therefore from (13.8.34)

\[
\nabla \cdot \int_0^{D_E} \rho \vec{\tau}_e \, dz = \frac{1}{R} \nabla \cdot (R \times \vec{\tau}_0)
\]

(13.8.35)

Since we have assumed \( \rho = \text{const.} \) in the boundary layer we find from the continuity equation

\[
\nabla \cdot \int_0^{D_E} \rho \vec{v}_e \, dz = -\rho W_F
\]

(13.8.36)

where \( W_F \) is the vertical velocity at the level \( z = D_E \).

Substituting in (13.8.35) we find

\[
W_F = \frac{1}{\rho_F} \nabla \cdot (R \times \vec{\tau}_0)
\]

(13.8.37)

which says that the frictionally induced vertical velocity is proportional to the vertical component of the curl of the stress. We may get additional information about this by introducing the expression

\[
\vec{\tau}_o = \rho c_d \, \vec{v}_o \times \vec{v}_o
\]

(13.8.38)

obtained from (13.6.20). Assuming for simplicity that the horizontal variation of
the wind speed is small compared with the spatial variation of the components we get

$$\nabla \cdot \tau_o \approx \rho d V_o \xi_o \quad (13.8.39)$$

or

$$W_p = \frac{c_d V_o}{f} \xi_o \quad (13.8.40)$$

$W_p$ is thus proportional to the surface vorticity. Using $c_d = 2.4 \times 10^{-3}$, $V_o = 5 \text{ m s}^{-1}$ we find for $|\xi_o| = f$ that $W_p \approx 1.2 \text{ cm s}^{-1}$. Because $\xi_o$ is in general smaller than $f$ we will most often get values of $W_p$ smaller than this estimate.

The result given in (13.3.37) and in approximate form in (13.8.40) does not depend on the details of the Ekman-solution. It can in fact be derived easily by simply noting that we have a balance between the pressure force, the Coriolis force and the frictional force in the layer. Expressing this in pressure co-ordinates we get

$$0 = -\nabla \psi - f \tau \times \vec{\nabla} - g \frac{d \tau}{dp} \quad (13.3.41)$$

equivalent to (13.3.1). The vorticity equation corresponding to (13.3.41) considering $\vec{\nabla} = \text{const.}$ is:

$$0 = f \nabla \cdot \vec{\nabla} - g \frac{d}{dp} (\nabla \cdot \tau) \quad (13.3.42)$$

Introducing the continuity equation $\nabla \cdot \vec{\nabla} = -\partial \omega / \partial p$ and integrating from $p_T$, the top of the Ekman-layer where $\tau \approx 0$, to $p_s$, where $\omega \approx 0$, we find

$$0 = -f \omega_p - g \nabla \cdot \vec{\tau}_o \quad (13.3.43)$$

or

$$\omega_p = -\frac{g}{f} \nabla \cdot \vec{\tau}_o \quad (13.3.44)$$

corresponding to (13.3.37).
If we want to use the details of the Ekman solution, we may use (13.3.35) and express $\vec{v}_0$ using the form

$$
\vec{v}_0 = \rho K \left( \frac{\partial \vec{v}}{\partial z} \right)_0 = \rho K \left\{ \vec{v}_0 - (u_0 - V_g) \right\} \vec{r} - \left\{ \vec{v}_0 + (u_0 - V_g) \right\} \vec{j} \tag{13.3.45}
$$

Substituting from (13.3.27), we find, after elementary trigonometric calculations and after introducing (13.3.15), that

$$
\vec{v}_g = \sqrt{2\pi K} \sin a_0 \left\{ \vec{v}_g \cos a_0 + \vec{r} \times \vec{v}_g \sin a_0 \right\} \tag{13.3.46}
$$

because $\vec{v}_g = \vec{v}_g$ and $\vec{v}_g = \vec{r} \times \vec{v}_g$.

(13.3.46) is introduced in (13.3.34), and we get

$$
\int_{\rho \vec{v}}^{D_E} dz = (D_E - H_\ast \sin^2 a_0) \vec{v}_g + H_\ast \sin a_0 \cos a_0 \vec{r} \times \rho \vec{v}_g \tag{13.3.47}
$$

from which

$$
\nabla \cdot \int_{\rho \vec{v}}^{D_E} dz = - \frac{1}{2} H_\ast \sin 2a_0 \rho \zeta_g \tag{13.3.48}
$$

or, using the same method as before,

$$
W_F = \frac{1}{2} H_\ast \sin 2a_0 \zeta_g \tag{13.3.49}
$$

which shows how $W_F$ can be computed from the geostrophic vorticity $\zeta_g$. Using the same values as before we have $H_\ast \approx 316$ m ($K = 5 m^2 s^{-1}$, $f = 10^{-4}$ s$^{-1}$). For $a_0 = \pi/3$ we get

$$
W_F = 112 \zeta_g \tag{13.3.50}
$$

For $\zeta_g = 10^{-4}$ s$^{-1}$ we find $W_F = 1.1$ cm s$^{-1}$

13.9 **Turbulent Transfer of Heat**

Let us consider the first law of thermodynamics using the simplifying assumption of constant density. We have then $\rho = \text{const.}$ and

$$
\nabla \cdot \vec{v} + \frac{dW}{dz} = 0 \tag{13.9.1}
$$
The first law of thermodynamics may be written in terms of the potential temperature as

\[
\frac{d\theta}{dt} + \nabla \cdot (\theta \mathbf{v}) + \frac{\partial (\theta w)}{\partial z} = \frac{1}{c_p} \frac{\theta}{T} \frac{\partial \theta}{\partial z} \tag{13.9.2}
\]

where already we have made use of (13.9.1). Defining the time average by a bar and the deviation by a prime, we get

\[
\frac{d\bar{\theta}}{dt} + \nabla \cdot \left( \bar{\theta} \mathbf{v} + \theta' \mathbf{v}' \right) + \frac{\partial \bar{\theta} w}{\partial z} + \frac{\partial \theta' w'}{\partial z} = \frac{1}{c_p} \frac{\bar{\theta}}{T} \frac{\partial \bar{\theta}}{\partial z} \tag{13.9.3}
\]

or

\[
\frac{d\bar{\theta}}{dt} = - \nabla \cdot \bar{\theta} \mathbf{v} - \bar{\theta} \frac{\partial \bar{\theta}}{\partial z} - \frac{\partial \theta' w'}{\partial z} + \frac{1}{c_p} \frac{\bar{\theta}}{T} \frac{\partial \bar{\theta}}{\partial z} \tag{13.9.4}
\]

where, in agreement with the earlier treatment, we have neglected the contributions from the horizontal convergence of the turbulent transfer of heat. In agreement with the mixing length hypothesis we have

\[
\bar{\theta}' w' = - K_H \frac{\partial \bar{\theta}}{\partial z} \tag{13.9.5}
\]

and if we wish to isolate the effect of the vertical turbulent transfer of heat, we get

\[
\frac{d\bar{\theta}}{dt} = \frac{d}{dz} \left( K_H \frac{d\bar{\theta}}{dz} \right) \tag{13.9.6}
\]

(13.9.6) looks rather simple because it has one dependent variable only. Indeed, it would be straightforward to solve if \( K_H \) were a constant, and even if \( K_H \) were a specified function of \( z \). However, the truth of the matter is that (13.9.6) is a non-linear equation since \( K_H \) is a function of \( \partial \bar{\theta}/\partial z \). Given this condition it is understandable that little is known in terms of analytic and realistic solutions. We shall be content with some qualitative considerations.

Let us first consider a case of instability (\( \partial \bar{\theta}/\partial z < 0 \)). We have then according to (13.9.5) that \( \theta' w' > 0 \), i.e. an upward turbulent transfer of heat. This is a case known as free convection. How can such conditions be generated? Typically they are created by heating from below. In such a case \( K_H \) is large, creating intense mixing, and it is to be expected that the potential temperature will tend to be a constant through the layer, i.e. the lapse rate will become adiabatic, except in the
lowest layer where $K_H$ is small since $w$ is small. Here the lapse rate is superadiabatic. How far the heating from below will be influencing the lower layers of the atmosphere can be determined if we know the initial temperature distribution with height and the amount of heat added from below.

Suppose that an amount $Q$ (energy per unit area) is added. The temperature will then increase by a certain amount $T' = T_F - T_I$ where $T_I$ is the initial and $T_F$ the final temperature. If we assume that the heat is influencing the temperature up to a height $z$ where $T' = 0$, we have

$$Q = \int_{c}^{z} c_p T' \, \rho \, dz = \frac{c_p}{\varepsilon} \int_{T_i}^{T_f} dp$$

or

$$\int_{p_T}^{p_S} \frac{c_p}{\varepsilon} \frac{dp}{dp} = \frac{c_p}{\varepsilon} Q = A$$

(13.9.7)

(13.9.8)

![Figure 13.9](image)

Figure 13.9 shows the initial temperature distribution, marked I, in a $(p, T)$ diagram, where we have assumed a linear curve for simplicity. The curve F shows the final distribution after heating by the amount $Q$. The integral on the left side of (13.9.8) is the area of the triangle $T_{IS} T_{FS} T_T$ in Figure 13.9.
We have thus

\[ A = \frac{1}{2} (T_{PS} - T_{IS}) (p_s - p_T) = \frac{E}{c_p} Q \]  \hspace{1cm} (13.9.9)

We shall now calculate how far the heating will penetrate for a given value of \( Q \). The initial temperature distribution is

\[ T = T_{IS} + \left( \frac{dT}{dp} \right)_I (p - p_s) \]  \hspace{1cm} (13.9.10)

and the final distribution is

\[ T = T_{PS} + \left( \frac{dT}{dp} \right)_P (p - p_s) \]  \hspace{1cm} (13.9.11)

In these expressions \( (dT/dp)_I \) is an initial lapse rate \( \Gamma_I \) in the \( p \)-system as follows

\[ \Gamma_I = \left( \frac{dT}{dp} \right)_I = - \frac{1}{g_p} \left( \frac{dT}{dz} \right)_I = \frac{\gamma_I}{g_p} \approx \tilde{\gamma}_I \ (p = 10^{-3}) \]  \hspace{1cm} (13.9.12)

where \( \tilde{\gamma}_I \) is the ordinary lapse rate measured in deg 100 m^{-1}

Similarly

\[ \Gamma_P = \frac{\gamma_P}{g_P} = \tilde{\gamma}_P = 1 \]  \hspace{1cm} (13.9.13)

The two temperatures in (13.9.10) and (13.9.11) are equal at \( p = p_T \). We get therefore

\[ T_{PS} - T_{IS} = \left( 1 - \Gamma_I \right) (p_s - p_T) \]  \hspace{1cm} (13.9.14)

but using (13.9.9) we find

\[ 2 \frac{E}{c_p} \frac{Q}{p_s - p_T} = \left( 1 - \Gamma_I \right) (p_s - p_T) \]  \hspace{1cm} (13.9.15)

or

\[ p_s - p_T = \sqrt{\frac{2 \frac{E}{c_p} \frac{Q}{1 - \Gamma_I}}{}} \]  \hspace{1cm} (13.9.16)
It is seen that \((p_s - p_T)\) will be large if \((1 - \Gamma_I)\) is small, i.e. if the initial lapse rate is close to the adiabatic lapse rate, while \((p_s - p_T)\) will be quite small if \(\Gamma_I\) is close to the averaged lapse rate, say \(\Gamma_I = 0.5\). Suppose for example that we add heat at the rate 5 \(\text{Wm}^{-2}\) = 5 \(\text{j} \cdot \text{m}^{-2} \cdot \text{s}^{-1}\) = 432 \(\text{Kjm}^{-2} \cdot \text{day}^{-1}\). We find then that \(p_s - p_T = 42 \text{mb}\) if \(\Gamma_I = 0.5\), while we get the value \(p_s - p_T = 94 \text{mb}\) if \(\Gamma_I = 0.9\), when the unit for \(\Gamma\) is deg./(10mb).

We have thus shown that the heating will penetrate quite extensively for a given value of \(Q\) if the initial lapse rate is close to the dry-adiabatic lapse rate, but only a short distance if the initial lapse rate is relatively stable. The kind of estimate given here can be used to calculate the effects of diurnal heating over land or heating experienced by an air mass moving from a cold continent to a warm ocean in winter or from a warm continent to a colder ocean in summer.

Let us next consider the case of cooling from below, under stable stratification \((\delta \bar{\theta}/\partial z > 0)\) conditions indicating \(\bar{\theta}^{'w'} < 0\). If conditions are calm or with only weak winds there will be little turbulence under stable conditions. Cooling at the ground will then influence a shallow layer only where the temperature decreases and an inversion forms. The height of the inversion will increase very slowly if the cooling continues. The conditions above are typical of inversions formed during night-time under calm, continental conditions where the cooling is created by out-going long-wave radiation. The inversions will normally disappear in day-time because of the heating due to the incoming shortwave radiation, but in high latitudes during winter there may not be sufficient incoming radiation to dissolve the inversion. The cold air below the inversion may create winds if the terrain is sloping. For large slopes, such as at the edge of the ice cap of Greenland, there can be considerable wind speeds in these winds. Inversions are formed whenever there is sufficient cooling below in a stable air mass, such as a warm air mass moving out over colder water or in over an ice-covered or snow-covered region. Because of the cooling in the lowest layers we will normally have formation of dew on the ground, and if this process continues we may have a transport of moisture downwards. However, most often we have a decrease of temperature in the inversion layer, and it is seen that the formation of fog is likely.

If the winds are reasonably strong under stable stratification \(\bar{\theta}^{'w'} < 0\), there will be sufficient mechanical turbulence to create a fair amount of mixing in the lowest layers, and there will be a tendency to equalization of the potential temperature. If we have cooling from below, we will under these conditions decrease the temperature in the whole layer. If a layer of less mechanical turbulence exists
higher up in the atmosphere we may get an inversion layer formed as indicated in Figure 13.10.

![Diagram](image)

Figure 13.10

The discussion in this section has been of a qualitative nature. It is naturally possible to solve (13.9.6) in analytical form using simple assumptions regarding $K_n$ and its dependence on height, but we shall abstain from giving these formal solutions here.
Chapter XIV

SOME ASPECTS OF THE GENERAL CIRCULATION

14.1 Introduction

The general circulation of the atmosphere is in the broadest sense a complete description of the average atmospheric flow on the earth. Such a description is usually made in statistical terms, and these statistics are generated from the ensemble of daily flow patterns. The averages employed in the descriptions are both spatial and time averages (such as the zonal westerlies and easterlies for the year or for a particular month, or the normal map for a given time of the year). However, in addition to such time and space averages of wind, temperature, pressure and humidity we also include the variation of these quantities through the seasons in the general circulation.

During the last three decades it has become customary to include in the general circulation the description of atmospheric transport processes, especially the meridional (south-north) transport of such quantities as the angular momentum, the sensible heat, water vapour, etc. While special emphasis is put on the meridional transports of atmospheric quantities because of their importance for the maintenance of the zonally averaged fields, it is naturally also possible to consider atmospheric transports in other directions.

Included in the general circulation we find also a description of the energy relations in the atmosphere. Such a description will normally include not only the amount of energy and its seasonal variation but also a description of the most important generations, conversions and dissipations of atmospheric energy.

The descriptions of the many quantities mentioned above are far beyond the scope of this compendium. We shall therefore restrict ourselves to the more basic aspects of the problem.

14.2 Atmospheric Averages and Transports

Let us consider an arbitrary quantity \( b = b(\lambda, \varphi, p, t) \) in the atmosphere. We shall assume that data giving the time and space variations of \( b \) are available.
Of the space averages used in the general circulation we shall mention the area average

\[ b_S = \frac{1}{S} \int_S b \, dS \]  

(14.2.1)

where \( S \) is the total area of the region under consideration and \( dS \) is the area element. We mention in particular the area average over a fraction of the sphere. The area element is

\[ dS = a^2 \cos \varphi \, d\lambda \, d\varphi \]  

(14.2.2)

as seen from Figure 14.1

![Diagram of area element](image)

Figure 14.1

We may consider, as a special case of (14.2.1), the area between two latitude circles \( \varphi_1 \) and \( \varphi_2 \) \((\varphi_1 < \varphi_2)\). We then have

\[ S = 2a^2 (\sin \varphi_2 - \sin \varphi_1) \]  

(14.2.3)

and (14.2.1) becomes

\[ b_S = \frac{1}{\sin \varphi_2 - \sin \varphi_1} \int_{\varphi_1}^{\varphi_2} \int_0^{2\pi} b \cos \varphi \, d\lambda \, d\varphi \]  

(14.2.4)

(14.2.4) becomes the average over the whole globe when \( \varphi_1 = -\pi/2 \) and \( \varphi_2 = +\pi/2 \).

The space average most often used in general circulation studies is the zonal average, i.e. the average of \( b \) along a latitude circle, defined as follows

\[ b_z = \frac{1}{2\pi} \int_0^{2\pi} b \, d\lambda \]  

(14.2.5)
The zonal average is a function of $\varphi$, $p$ and $t$, but not of $\lambda$, since the $\lambda$-dependence has been averaged out. The deviation of $b$ from the zonal average $b_z$ is called the eddy component, and it is defined by the relation

$$b_E = b - b_z$$  \hfill (14.2.6)

The quantity $b_E$ describes what we commonly call the atmospheric waves on the synoptic maps. It follows immediately from (14.2.6) that $(b_E)_z = 0$. We shall in the following make use of the decomposition described by (14.2.5) and (14.2.6). In applications we very often meet the zonal average of a product of two scalar fields such as $(bc)_z$. We note once and for all that such a product can be written in the form

$$(bc)_z = b_z c_z + (b_E c)_z$$  \hfill (14.2.7)

The proof of (14.2.7) is obtained by noting that

$$(bc)_z = b_z c_z + (b_E c_z)_z + (b_z c_E)_z + (b_E c)_z$$  \hfill (14.2.8)

where the two terms in the middle vanish because $(b_E)_z = 0$ and $(c_E)_z = 0$.

With respect to time averages, there are many in use in general circulation studies. They are, however, all of the form

$$\overline{b} = \frac{1}{T} \int_0^T b \, dt$$  \hfill (14.2.9)

but they vary with respect to the magnitude of $T$, the averaging time. We shall here be satisfied with a consideration of an average over a "long-term" meaning an averaging period $T$ so large that, for practical purposes, $\overline{b}$ in (14.2.9) can be considered as independent of time. If $T$, for example, is of the order of magnitude of several years we can normally make the assumption stated above. The so-called normal maps for selected levels in the atmosphere are examples of time-averages constructed according to (14.2.9). We also have normal maps for each individual month. They are obtained using (14.2.9) for each month letting $T = 1$ month. However, out of the monthly averages we form a "grand" average which then becomes the normal map for the month.

We shall next consider atmospheric transports, also called atmospheric fluxes. If $\mathbf{v}$ is the three-dimensional velocity vector, we call by the transport
vector. We may naturally divide $\mathbf{b}$ into the three components $b_u$, $b_v$, and $b_w$, called the zonal, meridional, and vertical transports, respectively. The transport vector is defined in each point of the atmosphere. We are, however, very often interested in the total transport across a boundary of a region or across the surface. For example, if we want to calculate the total transport of $b$ across a latitude circle we find

$$\int_0^{2\pi} b_v \cos \varphi \, d\lambda = 2\pi a \cos \varphi \left( b_v \right)_z$$

(14.2.1c)

where the last expression is obtained using (14.2.5).

The purpose of this section has been to introduce the various space and time averages used in studies of the general circulation. These concepts and definitions will be used in the next section to study the maintenance of the zonal winds.

14.3 The Momentum Budget

The main problem in this section is the maintenance of the zonal averaged $u$-component of the wind. We start by a short description of the observed distribution of $u_z$. At the ground we find, in the long-term average, easterly winds in the low latitudes, westerly winds in the middle latitudes, and weak easterlies in the polar regions. The easterlies in the low latitudes continue to quite high elevations. The westerlies in the middle latitudes increase with height in agreement with the thermal wind equation. In order to analyse the process which governs the changes in the zonally averaged wind we shall employ the first equation of motion in spherical co-ordinates. We have

$$\frac{\partial u}{\partial t} + \frac{u}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{v}{a} \frac{\partial u}{\partial \varphi} + \omega \frac{\partial u}{\partial p} = - \frac{1}{a \cos \varphi} \frac{\partial \phi}{\partial \lambda} + f v + \frac{uv}{a} \tan \varphi + F_\lambda$$

(14.3.1)

We will also need the continuity equation in spherical co-ordinates

$$\frac{1}{a \cos \varphi} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial v \cos \varphi}{\partial \varphi} \right) + \frac{\partial \omega}{\partial p} = 0$$

(14.3.2)

Before we form the zonal average of (14.3.1) we shall rearrange the equation in the form

$$\frac{\partial u}{\partial t} + \frac{1}{a \cos \varphi} \frac{\partial (uv)}{\partial \lambda} + \frac{1}{a} \frac{\partial (uv)}{\partial \varphi} + \frac{\partial (uv)}{\partial p} -$$

$$\frac{u}{a \cos \varphi} \left( \frac{\partial u}{\partial \lambda} + \frac{1}{a} \frac{\partial v}{\partial \varphi} + \frac{\partial \omega}{\partial p} - \frac{\tan \varphi}{a} v + \frac{\tan \varphi}{a} \nu \right)$$

$$- \frac{u}{a \cos \varphi} \frac{\partial \phi}{\partial \lambda} + f v + \frac{uv}{a} \tan \varphi + F_\lambda$$

(14.3.3)
We notice that the first four terms in parenthesis on the left-hand side of (14.3.3) add up to zero. Collecting and rearranging the remaining terms we get

\[ \frac{\partial u}{\partial t} + \frac{1}{a \cos \varphi} \frac{\partial (uv)}{\partial \lambda} + \frac{1}{a \cos^2 \varphi} \frac{\partial (uv) \cos^2 \varphi}{\partial \varphi} + \frac{\partial (\omega)}{\partial p} = \]

\[ - \frac{1}{a \cos \varphi} \frac{\partial \Phi}{\partial \lambda} + f v + F_{\lambda} \] \quad (14.3.4)

It is now a straightforward matter to form the zonal average of (14.3.4). We shall first assume that the pressure level under consideration is such that the latitude circle is entirely in the atmosphere. This is not always true at low elevations. Under the assumption above we get

\[ \frac{\partial u_z}{\partial t} = - \frac{\partial (uv)_z \cos^2 \varphi}{a \cos^2 \varphi \partial \varphi} - \frac{\partial (\omega)_z \cos^2 \varphi}{\partial p} + f v_z + F_{\lambda,z} \] \quad (14.3.5)

Equation (14.3.5) says that the zonally averaged u-component can change due to four terms on the right-hand side of (14.3.5). The first term is the horizontal convergence of the meridional transport of the u-momentum, while the second term is the vertical convergence of the vertical transport of u-momentum. The third term is the effect of the Coriolis term, while the last term is the frictional effect.

Figure 14.2 shows the kinematic processes which determine the magnitude of the first two terms in (14.3.5). The small square marked 1, 2, 3, 4 is the
cross-section of a zonal ring. The first term measures the difference between what comes in through the southern boundary (surface 1) and goes out through the northern boundary (surface 3). It is for this reason that we call the term the convergence of the meridional transport of momentum. The second term measures in a similar way the difference between the vertical transport through the top surface 2 and the bottom surface 4. $u_z$ will increase according to (14.3.5) if we have convergence of either the meridional or the vertical transport of momentum or both. Furthermore, $u_z$ will increase if $v_z > 0$ and decrease if $v_z < 0$.

Let us next consider the case where the latitude circle goes through one or several mountains. In that case we will not get a vanishing contribution from the pressure force and the second term on the left-hand side of (14.3.4).

![Figure 14.3](image)

The situation in the case of a single mountain is illustrated in Figure 14.3. The contribution from the pressure force term is

$$-\frac{1}{a \cos \varphi} \left( \frac{\partial \phi}{\partial \lambda} \right)_z = \frac{1}{2 \pi a \cos \varphi} (\phi_{EA} - \phi_{WE})$$

(14.3.6)

where $\phi_{EA}$ is the geopotential on the east side of the mountain, and $\phi_{WE}$ is the geopotential on the west side as shown in Figure 14.3. If the latitude circle goes through several mountains there will be a contribution of the type (14.3.6) from each mountain.

We get by a similar consideration the following contribution from the second term in (14.3.5)

$$-\frac{1}{a \cos \varphi} \left( \frac{\partial u^2}{\partial \lambda} \right)_z = \frac{1}{2 \pi a \cos \varphi} (u_{EA}^2 - u_{WE}^2)$$

(14.3.7)

where we again must add up the contribution of the type (14.3.7) from each mountain along the latitude circle.
In order to simplify the discussion as much as possible we shall introduce a vertical average in addition to the zonal average. It is done by the definition

\[ b_N = \frac{1}{p_s} \int_0^{p_s} b \, dp \]  

(14.3.8)

Let us now adopt the simplified boundary conditions

\[ \omega = 0 \text{ at } p = 0 \text{ and } \omega = p \]  

(14.3.9)

We find then from (14.3.2) by a zonal average

\[ \frac{\partial v_z \cos \varphi}{a \cos \varphi \partial \varphi} + \frac{\partial \omega_z}{\partial p} = 0 \]  

(14.3.10)

and then by a vertical average

\[ \frac{\partial v_z m \cos \varphi}{\partial \varphi} = 0 \]  

(14.3.11)

from which it follows that \( v_z m \cos \varphi = \text{const.} \), but the constant must be zero since the equation also has to be satisfied at the North and South Poles. Therefore \( v_z m = 0 \).

In addition, we shall assume that \( F = + g \partial \tau / \partial p \). Taking a vertical average of (14.3.5) under these conditions, we get

\[ \frac{\partial u_z}{\partial t} = - \frac{\partial (uv) z m \cos^2 \varphi}{a \cos^2 \varphi \partial \varphi} + \frac{g}{p_s} \tau_{x,\lambda, z, s,} \]  

(14.3.12)

if we use the condition that the stress vanishes at the top of the atmosphere and if for the moment we disregard the mountain effects given in (14.3.6) and (14.3.7). For \( \tau_{x, \lambda, z, s,} \), the zonal average of the zonal component of the surface stress, we may think of an expression of the nature

\[ \tau_{x, \lambda, z, s,} \approx - c_d \rho_s V_s u_{z s} \]  

(14.3.13)

where \( c_d \) is the drag coefficient, \( \rho_s \) the surface density and \( V_s \) the surface wind speed. It is thus obvious that the last term in (14.3.12) is negative in regions where \( u_{z s} > 0 \) and positive in regions where \( u_{z s} < 0 \). We may therefore say that the effect of the surface stress is to decrease the u-momentum in the westerlies and increase it in the easterlies. The low (and very high) latitudes are therefore
sources of westerly momentum due to the action of the surface stress, while the middle latitudes are a sink of westerly momentum. However, in the average, the westerly momentum remains constant over a long period, i.e. \( \bar{\frac{\partial u_z}{\partial t}} = 0 \), where the bar means a time average over a long period. The two terms on the right-hand side of (14.3.12) will therefore keep each other in balance, and we may say that the convergence of the momentum transport has the opposite sign of the surface stress term. We must therefore have convergence of the meridional transport of momentum in the middle latitudes and divergence in the low latitudes and, perhaps, the very high latitudes.

The result just obtained depends upon the distribution of the surface winds and the validity of (14.3.13). It can, however, be checked by a direct calculation of the momentum transport \((uv)_z\) from observed winds or geostrophic winds. Such calculations have been made by many investigators. It turns out that the results of these studies all agree on the shape of the curve \((uv)_z\) as a function of latitude. A schematic curve for \((uv)_z\) is shown in Figure 14.4. It shows an increase in the momentum transport from small values at the equator to a maximum at \(30^\circ N\), approximately; a decrease from there to a negative minimum around \(75^\circ N\) and then an increase to zero at the North Pole. It is seen directly from the curve in Figure 14.4 that there will be divergence of the momentum transport south of \(30^\circ N\) and north of \(75^\circ N\), but convergence between these two latitudes as required by the distribution of the surface stress.

![Figure 14.4](image-url)
The results above were obtained disregarding the effects of the mountains as expressed in (14.3.6) and (14.3.7). We notice that each of the terms will disappear if the conditions are completely symmetrical around the mountains. This is however not the case as we shall see later. Both observational studies and theory show that \( \phi_{EA} < \phi_{WE} \) in regions of \( u_z > 0 \) while \( \phi_{EA} > \phi_{WE} \) in regions of \( u_z < 0 \). One finds similarly that \( u_{EA}^2 < u_{WE}^2 \) when \( u_z > 0 \), while the opposite holds when \( u_z < 0 \). One may therefore conclude from these investigations that the mountain effect acts in roughly the same way as the surface stress and that the statements made in connexion with (14.3.12) still hold.

We have here considered the balance requirements for \( u_z \), which are the most simple to discuss. It is seen from (14.3.5) that the Coriolis effect \( f v_z \) and the vertical convergence of the vertical momentum transport, i.e. \( \partial (u \omega)_z / \partial p \), must be considered if we want to consider the balance requirements at a specified pressure level in the atmosphere. However, it will be necessary to discuss the mean meridional circulation before we can give an evaluation of these terms.

The mean meridional circulation is the circulation with the velocity components \( v_z \) and \( \omega_z \). It can be described entirely in a meridional cross-section. \( v_z \) and \( \omega_z \) are not independent of each other, but are connected through the zonal average of the continuity equation, i.e.

\[
\frac{1}{a \cos \phi} \frac{\partial v_z \cos \phi}{\partial \phi} + \frac{\partial \omega_z}{\partial p} = 0
\]  

(14.3.14)

\( v_z \) can be computed directly from observed winds if we have accurate wind observations in a dense network, but it cannot be obtained from geostrophic winds because

\[
v_{g,z} = \frac{1}{a \cos \phi} \left( \frac{\partial \Phi}{\partial \lambda} \right)_z = 0
\]  

(14.3.15)

Since the atmosphere at any time contains waves, it is easily verified that \( v_z \) must be considerably smaller than \( v_{g,z} \) because of the compensation in the evaluation of \( v_z \) between the negative values of \( v \) between the ridge and the trough and the positive values of \( v \) between the trough and ridge looking downstream. The compensation just mentioned makes a direct calculation of \( v_z \) from observed winds very inaccurate unless extreme care is taken in the handling of the data. Direct calculations have nevertheless been attempted, but it has also been shown that the results are quite sensitive to the methods of analysis. For our purpose it is sufficient to mention that the various direct calculations of \( v_z \) from various analyses agree in the essential features which we shall make use of.
The inaccuracies connected with the direct determination of $v_z$ have also given impetus to the design of indirect calculations of $v_z$ or $\omega_z$. One may for example attempt to solve the $\omega$-equation (see section 12.6) in a zonally averaged form from which one obtains $\omega_z$ and then $v_z$ from (14.3.14). It is not possible here to discuss the various indirect ways of obtaining the mean meridional circulation, but we stress that the results are in qualitative agreement with those obtained by direct methods. The most convenient way of expressing the mean meridional circulation is probably through a stream function in the meridional plane. Such a function can be defined in several ways, but it is most common to use a stream function with the dimension of a massflux, i.e. $g \text{ s}^{-1}$ or $t \text{ s}^{-1}$. We then get

$$v_z = \frac{E}{2\pi a^2} \frac{a}{\cos \varphi} \frac{\partial \psi}{\partial p} \quad \text{(14.3.16)}$$

$$\omega_z = -\frac{E}{2\pi a^2} \frac{1}{\cos \varphi} \frac{\partial \psi}{\partial \varphi}$$

It is seen, by substitution of (14.3.16) in (14.3.14), that the latter equation is indeed satisfied. The results of the direct and indirect methods are given in schematic form in Figure 14.5. Please note that Figure 14.5 is highly idealized and that actual calculations give more irregular features of the meridional stream function, but the essential features of all the calculations are included in the figure which shows three cells. The cell closest to the equator is called the Hadley cell because it was originally postulated by Hadley from thermodynamic considerations in an attempt to explain the existence of the trade winds. The cell is also called a thermodynamically direct cell because the warm (and light) air at the equator is rising while the somewhat colder air at $30^\circ \text{N}$ is sinking. Such a process will bring the centre of gravity to a lower position and therefore decrease the potential energy and, normally, increase the kinetic energy.

![Figure 14.5](image-url)
The cell in the middle latitudes is called the Ferrel cell. It is thermally indirect because the sinking air at 30°N is warmer than the rising air at 60°N. The circulation will increase the potential energy and decrease the kinetic energy of the mean meridional motion. The last cell is the polar cell. It is also a thermally direct cell.

The three cells do not have the same intensity. The most well-developed cell is the Hadley cell, especially in winter. The Ferrel cell exists throughout the year but is normally of smaller intensity than the Hadley cell. The polar cell is the least developed of the three cells, and it may even be missing altogether.

In order to describe the relative roles of the mean meridional circulation and the eddies we shall now return to (14.3.5) and introduce that

\[(uv)_z = u_z v_z + (u_E v_E)_z\]

and

\[(u\omega)_z = u_z \omega_z + (u_E \omega_E)_z\]  \hspace{1cm} (14.3.17)

We then get

\[
\frac{\partial u_z}{\partial t} = -\frac{\partial u_z v_z \cos^2 \varphi}{a \cos^2 \varphi \partial \varphi} - \frac{\partial u_z \omega_z}{\partial \varphi} + f v_z + F_{\lambda z} - \frac{\partial (u_E v_E)_z \cos^2 \varphi}{a \cos^2 \varphi \partial \varphi} - \frac{\partial (u_E \omega_E)_z}{\partial \varphi} \hspace{1cm} (14.3.18)
\]

Differentiating out in the first two terms on the right-hand side of (14.3.18) and using the continuity equation (14.3.14), we get

\[
\frac{\partial u_z}{\partial t} = \left( f - \frac{1}{a \cos \varphi} \frac{\partial u_z \cos \varphi}{\partial \varphi} \right) v_z - \omega_z \frac{\partial u_z}{\partial \varphi} + F_{\lambda z} - \frac{\partial (u_E v_E)_z \cos^2 \varphi}{a \cos^2 \varphi \partial \varphi} - \frac{\partial (u_E \omega_E)_z}{\partial \varphi} \hspace{1cm} (14.3.19)
\]

The first two terms in (14.3.9) express the effects of the mean meridional circulation on the $u_z$-field, while the effects of the eddies are expressed by the last two terms. The very first term contains the product of the absolute vorticity and the mean meridional velocity $v_z$. The sign of the term will be the same as the sign of $v_z$ because the absolute vorticity is positive except for the very low latitudes.
It is therefore very easy to get the sign of the term by using the distribution shown in Figure 14.5. Using the same figure one may equally well obtain the sign of the second term on the right-hand side of (14.3.19).

The evaluation of the various terms in (14.3.19) can answer the question of the relative importance of the mean meridional circulation and the eddies in maintaining the zonal winds. The outcome of the investigations is that the eddies play the most important role by far in the middle and high latitudes, while the mean meridional circulation, i.e. the Hadley circulation, has the dominant role in the low latitudes. These conclusions have been reached by comparing the horizontal processes, while less is known about the effects of the vertical transport terms except that in general these terms are much smaller than their horizontal counterparts. We stress that the conclusion applies to the vertical mean as expressed in (14.3.12).

In view of the conclusion given above it is important to investigate the structure of the waves which outside the tropics are so important in carrying out the necessary momentum transport to balance the destruction of momentum by the frictional forces. It turns out that it is the shape of the waves which is the most important factor. Let us consider a geopotential of the simple mathematical form

$$\phi = A \cos (kx - ay)$$

(14.3.20)

For each value of y we have a sinusoidal wave in the geopotential. The ridge line of the wave is located at \(x = 0\) for \(y = 0\). For any other value of \(y\) we find the ridge line at the position \(x_m = \frac{a}{k} y\). If \(a\) is positive therefore we have a wave where the ridge line (and the trough line) slope from south-west to north-east, while \(a < 0\) implies a slope from north-west to south-east. In the special case of \(a = 0\) we have a wave with a south-north direction of the ridge and trough lines. We shall now evaluate the geostrophic-momentum transport in the wave. We find

$$u = -\frac{A}{f} a \sin(kx - ay)$$

(14.3.21)

and

$$v = -\frac{A}{f} k \sin(kx - ay)$$

(14.3.22)

and therefore

$$(u v)_{z} = \frac{1}{2} \frac{A^2}{f^2} ka$$

(14.3.23)
where the mean value in this case is taken over one wavelength \( L = 2\pi/k \). It is therefore obvious from (14.3.23) that waves with \( a > 0 \) will transport momentum to the north, while the opposite transport takes place when \( a < 0 \). Figure 14.6 and Figure 14.7 show waves with \( a > 0 \) and \( a < 0 \) respectively. It is seen from Figure 14.6 (\( a > 0 \)) that the negative contributions to \( (u_E^2) \) from the region where \( v_E < 0 \), such as in point A are smaller than the positive contributions from the region where \( v_E > 0 \), as in point B. It is thus geometrically obvious that the net contribution from the wave pictures in Figure 14.6 is positive. A similar analysis of the wave in Figure 14.7 shows that the momentum transport \( (u_Ev_E)_z < 0 \). Comparing now the
results in (14.3.23), Figure 14.6 and Figure 14.7 with the curve given in Figure 14.4 when the contribution outside the tropics comes from the eddies, we may conclude that the atmospheric waves in the region from about 25-30°N to about 60°N are, on the average, of the type shown in Figure 14.6, while the waves in the very high latitudes are more often than not of the type given in Figure 14.7. An inspection of a long series of synoptic charts will confirm this conclusion.

It was pointed out that the dominant role of the eddies in the required momentum transport in middle and high latitudes applies to the total transport or the vertical mean. A similar statement cannot be made if we consider an individual level. The mean meridional circulation may be very essential in maintaining the zonal winds at that level. As an example, let us consider a point in the middle latitudes and at a low elevation. Most of the momentum transport takes place at high altitudes. The contribution from the convergence of the momentum transport is, therefore, of minor importance at low elevations. The balance here is mainly between the contribution from η_zv_z, where

$$\eta_z = \frac{1}{a \cos \varphi} \frac{\partial u \cos \varphi}{\partial \varphi}$$  \hspace{1cm} (14.3.24)$$

and where

$$\eta_zv_z > 0$$ and $$F_{\lambda,z}$$, which is negative in middle latitudes.

On the other hand, at high elevations and in middle latitudes we have a very large positive contribution from the convergence of the momentum transport due to the eddies, but little or no contribution from friction. The negative contribution which makes the balance comes in this case from $$\eta_zv_z$$ which is negative as seen from Figure 14.5. A similar analysis can easily be carried out in the very high latitudes.

We have in this section preferred to discuss the zonal average of the first equation of motion and thereby to consider the maintenance of the $$u_z$$-field. It should be mentioned that the whole discussion can also be carried out in terms of the angular momentum, which is introduced in the following way. Consider a particle with the relative zonal velocity u. It will have the absolute velocity: $$Ω_0 \cos \varphi + u$$ if the particle is at latitude ϕ. The momentum around the axis of the earth is (see Figure 14.8): $$(Ω_0 \cos \varphi + u) a \cos \varphi$$.
Figure 14.8

We consider finally the total angular momentum of all the particles at latitude $\varphi$. It is

$$M = \int_0^{2\pi} (\Omega a \cos \varphi + u) a \cos \varphi a \cos \varphi \, d\lambda$$  \hspace{1cm} (14.3.25)

or

$$M = 2\pi a^2 \cos^2 \varphi (\Omega a \cos \varphi + u_z)$$  \hspace{1cm} (14.3.26)

It is seen from this equation that

$$\frac{dM}{dt} = 2\pi a^2 \cos^2 \varphi \frac{du_z}{dt}$$  \hspace{1cm} (14.3.27)

We may therefore obtain the equations for the rate of change of the angular momentum $M$ by multiplying all of the equations for $\frac{du_z}{dt}$ by $2\pi a^2 \cos^2 \varphi$. 
14.4 Atmospheric Energy (an introduction)

The atmosphere contains energy of three main kinds: internal, potential and kinetic energy. Since we are treating the atmosphere as an ideal gas we know from thermodynamics that the internal energy per unit mass is: \( i = c_v T \), where \( c_v \) is the specific heat for constant volume. The potential energy per unit mass is \( \phi = gz \), where \( z \) in atmospheric studies is normally counted from mean sea-level. The kinetic energy per unit mass is \( k = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \) where \( \mathbf{v} \) is the three-dimensional velocity vector. In summary we have:

\[
i = c_v T, \quad \phi = gz, \quad k = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \quad (14.4.1)
\]

If the volume element is denoted \( dV \) and the density is \( \rho \), we find the amounts

\[
i \rho dV, \quad \phi \rho dV, \quad k \rho dV \quad (14.4.2)
\]

in the mass element \( \rho dV \). The integrals of the quantities listed in (14.4.2) over the total volume under consideration will give the total energy amounts. We get

\[
I = \int i \rho dV, \quad P = \int \phi \rho dV, \quad K = \int k \rho dV \quad (14.4.3)
\]

It is seen that the amounts of energy may be calculated if we know the physical variables and the field of motion. In addition to the amounts of energy, we are most often interested in the way in which energy changes in the atmosphere. We shall start out by considering the rate of change for a particle of unit mass.

We find for the internal energy that

\[
\frac{di}{dt} = c_v \frac{dT}{dt} = H - p \frac{da}{dt} \quad (14.4.4)
\]

where the last expression is obtained from the first law of thermodynamics. We may introduce the continuity equation in (14.4.4), and we get

\[
\frac{di}{dt} = H - p \mathbf{a} \cdot \mathbf{v} = H - RT \mathbf{v} \cdot \mathbf{v} \quad (14.4.5)
\]

The rate of change of the potential energy is

\[
\frac{d\phi}{dt} = \frac{dz}{dt} = \phi \quad (14.4.6)
\]
The rate of change of the kinetic energy is

\[
\frac{dk}{dt} = \frac{u}{dt} + v \frac{dv}{dt} + w \frac{dw}{dt} \tag{14.4.7}
\]

and it is therefore seen that an expression for \( \frac{dk}{dt} \) can be obtained from the
general equations of motion (see (1.12.25)) by multiplying the first equation by \( u \),
the second by \( v \), the third by \( w \), and adding the resulting equations. We then get

\[
\frac{dk}{dt} = - \pi \cdot \pi v - \pi w + \pi \pi \cdot \pi \tag{14.4.6}
\]

We note that in deriving (14.4.8) from (1.12.25) there is no contribution
from the Coriolis terms. The Coriolis force gives no contribution because it is
always perpendicular to the velocity and for this reason cannot do any work. The
remaining terms in (1.12.25) are all due to the spherical co-ordinate system, and
they cannot give any contribution.

Addition of (14.4.5), (14.4.6) and (14.4.8) leads to the general equation
for the total energy

\[
\frac{d}{dt}(i + \phi + k) = - \pi \cdot (\pi v) + H + \pi \pi \cdot \pi \tag{14.4.9}
\]
or

\[
\rho \frac{ds}{dt} = - \pi \cdot (\pi v) + \rho H + \pi \pi \cdot \pi \tag{14.4.10}
\]

where \( \rho H \) is the heating per unit volume, and \( S = i + \phi + k \).

We note that

\[
\rho \frac{ds}{dt} = \phi \frac{dS}{dt} + \pi \cdot (\pi \pi) \tag{14.4.11}
\]

The proof of (14.4.11) is obtained by noting that

\[
\rho \frac{ds}{dt} = \phi \left( \frac{dS}{dt} + \pi \cdot \pi S \right) = \frac{dS}{dt} + \frac{\pi}{\pi} - \frac{\pi}{\pi} + \rho \pi \cdot \pi \tag{14.4.12}
\]

Substituting from the continuity equation for \( \frac{dp}{dt} \) we get finally

\[
\rho \frac{ds}{dt} = \phi \frac{dS}{dt} + \pi \cdot \pi + \pi \cdot (\pi v) \cdot \phi S \tag{14.4.13}
\]

from which (14.4.11) is obtained.
Using \((14.4.11)\) we may now write \((14.4.10)\) in the form

\[
\frac{\partial \hat{S}_p}{\partial t} = - \nabla \cdot \left[ \hat{S}_p \nabla \right] - \nabla \cdot \left[ \frac{RT \nabla}{\rho} \right] + \rho H + \nabla \cdot \vec{F} \tag{14.4.14}
\]

or

\[
\frac{\partial}{\partial t} \left[ \left\{ \frac{i + \phi + k}{\rho} \right\} \rho \nabla \right] = - \nabla \left[ \left\{ \frac{c_p T + \phi + k}{\rho} \right\} \rho \nabla \right] + \rho H + \nabla \cdot \vec{F} \tag{14.4.15}
\]

The first term on the right-hand side of \((14.4.15)\) is the convergence of the transports of sensible heat \((c_p T)\) (also called enthalpy), the potential energy and the kinetic energy. The equation above will be very important in the discussion of the atmospheric heat budget in section 14.6. For the moment we shall discuss certain integral aspects of \((14.4.15)\). Let us denote

\[
\vec{W} = \left\{ \frac{c_p T + \phi + k}{\rho} \right\} \rho \nabla
\]

We may then write \((14.4.15)\) in the form

\[
\frac{\partial \hat{S}_p}{\partial t} = - \nabla \cdot \vec{W} + \rho H + \nabla \cdot \vec{F} \tag{14.4.16}
\]

\(\vec{W}\) is a transport vector consisting of the transport of sensible heat, potential energy and kinetic energy. The term \((- \nabla \cdot \vec{F})\) is, according to \((14.4.8)\), the dissipation of kinetic energy due to actions of the frictional force. Note that \((- \nabla \cdot \vec{F}) > 0\). However, when kinetic energy is destroyed by friction we generate heat. We may therefore write

\[
\rho H_p = - \nabla \cdot \vec{F} \tag{14.4.17}
\]

On the other hand, by our definitions, \(H\) is the total amount of heating per unit mass including the frictional heating. Introducing

\[
H_{NF} = H - H_p \tag{14.4.18}
\]

we may call the quantity \(H_{NF}\) the non-frictional heating. With the notations \((14.4.17)\) and \((14.4.18)\) we may write \((14.4.16)\) in the form

\[
\frac{\partial \hat{S}_p}{\partial t} = - \nabla \cdot \vec{W} + H_{NF} \tag{14.4.19}
\]

where \(H_{NF}\) includes the heating due to all processes except friction.
Let us next integrate (14.4.19) over a volume in the atmosphere bounded by the surface of the earth, the top of the atmosphere, and two surfaces perpendicular to the earth along latitude circles (see Figure 14.9). The volume element is

\[ dV = a^2 \cos \phi \ d\lambda \ d\phi \ dz \]  

(14.4.20)

and we recall that

\[ \nabla \cdot \vec{V} = \frac{1}{a \cos \phi} \left( \frac{\partial W}{\partial \lambda} + \frac{\partial W_\phi \cos \phi}{\partial \phi} \right) \frac{\partial W_z}{\partial z} \]  

(14.4.21)

In the integration of (14.4.19) over the volume of the ring we shall use that the volume integral of the divergence is equal to the surface integral of the normal component of the vector.

![Figure 14.9](image)

Considering the four surfaces marked 1, 2, 3 and 4 in Figure 14.9 we note that there will be no contribution from surface 4 because the normal component vanishes due to the boundary condition. Similarly, there will be no contribution from surface 3 because here the normal component contains the factor \( p \) which goes to zero as \( z \to \infty \). We get therefore

\[ \int_V \frac{\partial \Phi}{\partial t} dV = \int_1 \int_{\phi_1}^{\phi_2} \int_{\lambda_1}^{\lambda_2} V_\phi \, d\sigma - \int_2 \int_{\phi_1}^{\phi_2} \int_{\lambda_1}^{\lambda_2} V_\phi \, d\sigma + \int_V H_{NP} \rho dV \]  

(14.4.22)

The area element is

\[ d\sigma = a \cos \phi \ d\lambda \ dz \]  

(14.4.23)
Furthermore, using the hydrostatic relation we may write (14.4.22) in the form

\[
\int \frac{\partial S}{\partial t} \, dV = \frac{1}{\mathcal{G}} \int \int \left\{ \left( c_p T + \phi + k \right)_1 \nu_1 \cos \varphi_1 d\lambda dp - \right.
\]

\[
\frac{1}{\mathcal{G}} \int \int \left\{ \left( c_p T + \phi + k \right)_2 \nu_2 \cos \varphi_2 d\lambda dp + \right. \]

\[
\int \frac{H_{NP}}{\mathcal{G}} \, dV \]

or

\[
\int \frac{H_{NP}}{\mathcal{G}} \, dV = \int \frac{\partial S}{\partial t} \, dV + \frac{\pi \mathcal{G} \cos \varphi_2}{\mathcal{G}} \int \int \left[ \left( c_p T + \phi + k \right) \nu_2 \right] \nu_2 \, d\lambda dp - \]

\[
\frac{\pi \mathcal{G} \cos \varphi_1}{\mathcal{G}} \int \int \left[ \left( c_p T + \phi + k \right) \nu_1 \right] \nu_1 \, d\lambda dp \]

This equation expresses the heat budget in a zonal ring of the type shown in Figure 14.9. The left-hand side of the equation expresses the total non-frictional heating in the zonal ring, while the first term on the right-hand side is the storage of energy. The last term is the divergence of the meridional flux of \((c_p T + \phi + k)\), i.e., the sum of the fluxes of the sensible heat, potential and kinetic energy. The equation will be most important in section 14.6 where we shall discuss the heat budget of such zonal rings. We note already that the storage term will tend to zero if we average (14.4.25) over a long period of time. This is because we have no systematic increase or decrease of the energy content anywhere in the atmosphere.

A second immediate use of (14.4.25) is to show that the total non-frictional heating of the atmosphere vanishes in the average over a long time period. We can see this from (14.4.25) by letting the zonal ring expand in such a way that it includes the whole atmosphere, i.e., \(\varphi_2 = \pi/2\) and \(\varphi_1 = -\pi/2\). The two-flux integrals will then become zero because of the cos-factors. Since the storage term also vanishes because of the averaging procedure, we find that the whole right-hand side of (14.4.25) will vanish in this case.

We recall that the non-frictional heating was defined in (14.4.18). For the whole atmosphere in the average for a long time period we find therefore that the total heating is equal to the heating due to frictional processes and therefore positive. Denoting the time average by a bar we have:
\[ H_{NP} = 0 \]  \hspace{1cm} (14.4.26)

and

\[ H = H_P > 0 \]  \hspace{1cm} (14.4.27)

We stress that the results just obtained are true only under the conditions stated. In particular, one cannot generally assume that the storage term vanishes over shorter time periods, and the flux terms are very important in the heat budgets of the zonal rings.

### 14.5 Global Aspects of Atmospheric Energy

The considerations in the preceding section were mainly devoted to the relation for the total energy (see equation (14.4.9)). We shall in this section consider the three different forms of energy separately and derive equations for the rate of change of each. To focus attention on the global aspects of the energy relations we shall consider the whole globe. The pertinent equations are (14.4.3) with the understanding that \( V \) is the volume of the total atmosphere. We find

\[
\frac{dI}{dt} = \int_V \frac{\partial (ip)}{\partial t} \, dV; \quad \frac{dp}{dt} = \int_V \frac{\partial (pp)}{\partial t} \, dV; \quad \frac{dk}{dt} = \int_V \frac{\partial (kp)}{\partial t} \, dV \quad (14.5.1)
\]

When we want to evaluate each of these integrals it is most convenient to use the following general relation

\[
\int_V \frac{\partial (bp)}{\partial t} \, dV = \int_V p \frac{db}{dt} \, dV \quad (14.5.2)
\]

which can be obtained by the use of (14.4.11) with \( S = b \); i.e. the proof follows identical lines. We then have

\[
\frac{\partial (bp)}{\partial t} = p \frac{db}{dt} - V \left[ bp \frac{\partial}{\partial t} \right] \quad (14.5.3)
\]

but the integral of the divergence term vanishes when \( V \) is the total atmosphere. It follows that

\[
\frac{dI}{dt} = \int_V p \frac{di}{dt} \, dV; \quad \frac{dp}{dt} = \int_V p \frac{dp}{dt} \, dV; \quad \frac{dk}{dt} = \int_V p \frac{dk}{dt} \, dV \quad (14.5.4)
\]
Using (14.4.5) we find

$$\frac{dI}{dt} = \int_V H \rho \, dV - \int_V pV \cdot \vec{V} \, dV$$

(14.5.5)

while (14.4.6) leads to

$$\frac{dP}{dt} = \int_V g \rho \omega \, dV$$

(14.5.6)

From (14.4.8) we obtain finally

$$\frac{dK}{dt} = -\int_V \vec{v} \cdot \nabla p \, dV - \int_V g \omega \rho \, dV + \int_V \vec{v} \cdot \vec{F} \, dV$$

(14.5.7)

Using the identity \( \vec{v} \cdot \nabla p = \nabla \cdot (p \vec{v}) - p \nabla \cdot \vec{v} \) and noting that \( \nabla \cdot (p \vec{v}) \) integrates to zero because \( V \) is the total volume of the atmosphere we get

$$\frac{dK}{dt} = \int_V p \vec{v} \cdot \vec{V} \, dV - \int_V g \omega \rho \, dV + \int_V \vec{v} \cdot \vec{F} \, dV$$

(14.5.8)

It is convenient to write the content of (14.5.5), (14.5.6) and (14.5.8) in a symbolic form. We note that the integral \( \int_V p \vec{v} \cdot \vec{V} \, dV \) appears in the equations for the internal energy and in the kinetic energy equation, but with opposite sign in the two equations. This means that the integral measures the energy transformation or the energy conversion between the internal energy and the kinetic energy. The symbolic way of writing such a term is

$$c(I,K) = \int_V p \vec{v} \cdot \vec{V} \, dV$$

(14.5.9)

where we have used the convention that the expression \( C(A,B) \) measures the energy conversion from \( A \) to \( B \). It follows immediately that

$$C(A,B) = -C(B,A)$$

(14.5.10)

Equations (14.5.6) and (14.5.8) show furthermore that

$$C(K,P) = \int_V g \rho \omega \, dV$$

(14.5.11)

The remaining terms in the three equations are the first term in (14.5.5) which measures the generation of internal energy by the heating of the atmosphere,
and the last term in (14.5.8), which measures the dissipation of kinetic energy by frictional processes. The convention is to denote the terms in the following way:

\[
G(I) = \int V_{\nu} dV \tag{14.5.12}
\]

and

\[
D(K) = -\int \nabla \cdot \mathbf{F} dV \tag{14.5.13}
\]

Note that it is customary to include the minus sign in (14.5.13) because we want to think of \(D(K)\) as a positive number. With these notations we may write (14.5.5), (14.5.6) and (14.5.8) in the symbolic forms

\[
\frac{dI}{dt} = G(I) - C(I,K)
\]

\[
\frac{dP}{dt} = C(K,P) \tag{14.5.14}
\]

\[
\frac{dK}{dt} = C(I,K) - C(K,P) - D(K)
\]

The generation, transformation and dissipation are shown schematically in the energy diagram in Figure 14.10. The dashed line in the figure shows that \(D(K)\) really is a conversion of kinetic energy into heat. At any given time we will normally have the tendencies, \(dI/dt\), \(dP/dt\) and \(dK/dt\), different from zero, and it is therefore pertinent to calculate all the processes entering the equations (14.5.14) and Figure 14.10. On the other hand, if we restrict ourselves to the average over a long period of time we will have \(dI/dt = dP/dt = dK/dt = 0\) because we do not have a systematic increase or decrease of any form of energy in the atmosphere.
It follows then that \( C(K, P) = 0 \) and that

\[
G(I) = C(I, K) = D(K) > 0
\]  
(14.5.15)

Each of the quantities in (14.5.15) may then be taken as a measure of the intensity of the general circulation of the atmosphere.

We shall now use the result, obtained in section 14.4, that

\[
H = H_F + H_{NF} = -\mathbf{a} \cdot \mathbf{F}
\]
(14.5.16)

and that

\[
\int_V H \rho dV = \int_V H_F \rho dV
\]
(14.5.17)

because

\[
G_{NF}(I) = \int_V H_{NF} \rho dV = 0
\]
(14.5.18)
It follows from these relations that
\[ G_P(I) = \int_H p_dV = - \int_v \vec{F}dV = D(K) \] (14.5.19)
or that the non-frictional heating generates no internal energy on the average over
a long period of time. Based on these results we can draw the diagram 14.10 in the
form given in Figure 14.11.

![Diagram](image)

Figure 14.11

It should be stressed that the conditions given in Figure 14.11 apply to the long-
term average only.

The energy equations (14.5.14) and the diagrams in Figure 14.10 and
Figure 14.11 are quite general because they are based on the fundamental equations.
However, as has been pointed out before, we always have the hydrostatic assumption
introduced in the routine data for the atmosphere. One of the implications of this
(very excellent) assumption is that the internal and the potential energies become
proportional when integrated over a vertical column. From the expression for the
potential energy in a vertical unit cross-section we get
\[ \int_{0}^{(g\Delta)z} d\zeta = \int_{0}^{\text{pdz}} \int_{0}^{\infty} \frac{R}{c_v} \int_{0}^{(c_vT)} \text{pdz} \] (14.5.20)
The integral appearing in the last part of (14.5.20) is the internal energy for the column, and the proportionality factor is therefore \( P/c_v \). Since the relation (14.5.20) is valid for each vertical column it is naturally also true for the whole atmosphere. We have therefore

\[
P = \frac{R}{c_v} I \tag{14.5.21}
\]

In view of (14.5.21) it becomes unnecessary to distinguish between the potential and the internal energies in a system employing the hydrostatic relation. It is therefore customary to replace the two energy reservoirs \( I \) and \( P \) by a single reservoir containing both the potential energy and the internal energy. We have

\[
E = P + I = \frac{R}{c_v} \int_0^\infty (c_v T) \rho dz + \int_0^\infty (c_v T) \rho dz - \int_0^\infty (c_p T) \rho dz \tag{14.5.22}
\]

The term \( E = P + I \) is often called the total potential energy. We can find the energy equation for \( E \) by adding the first two equations in (14.5.14).

We obtain

\[
\frac{dE}{dt} = C(I) - C(I,K) + C(K,P) \tag{14.5.23}
\]

\[
\frac{dK}{dt} = C(I,K) - C(K,P) - D(K)
\]

An inspection of (14.5.23) shows that it is natural to define

\[
C(E,K) = C(I,K) - C(K,P) = \int \vec{p} \cdot \vec{v} \, dV - \int \rho g w \, dV =
\]

\[
\int \vec{p}_2 \cdot \vec{v} \, dV - \int \rho \frac{\partial \vec{p}}{\partial z} \, dV - \int \rho g w \, dV =
\]

\[
\int \vec{p}_2 \cdot \vec{v} \, dV - \int_{\mathcal{S}} p_s w_s \, ds \tag{14.5.24}
\]

where the integral would vanish if there were no mountains on the earth, i.e. if \( w_s = 0 \). The content of (14.5.23) may be given in the form of an energy diagram as shown in Figure 14.12, where the term \( G(E) = G(I) \) and \( C(E,K) \) is given by (14.5.24).
The expression for $D(K)$ is the same as before, but it is customary to include only the horizontal wind because of the hydrostatic relation. In any case, we should recall that the vertical velocity in the hydrostatic system is computed from Richardson's equation for the vertical velocity (see Chapter III). Figure 14.12 shows that we may measure the intensity of the hydrostatic system by either $C(E,K)$ or $D(K)$. The possibility of using $G(E)$ cannot be used because only the frictional part of the heating will give a non-vanishing contribution, and the calculation of $G(E)$ is therefore equivalent to the evaluation of $D(K)$. It is naturally possible to derive the energy equations (14.5.23) directly from the equations after having incorporated the hydrostatic assumption. We shall do this, but for the p-system, because it is used in most investigations using atmospheric data. Starting from the thermodynamic equation we get

$$c_v \frac{dT}{dt} + p \frac{da}{dt} = c_p \frac{dT}{dt} - \frac{dp}{dt} = c_p \frac{dT}{dt} - \omega = H \quad (14.5.25)$$

We have the following expression for the rate of change of the total potential energy

$$\frac{dE}{dt} = c_p \int \int \int \frac{dT}{dt} p dS \; dz = c_p \int \int \int \frac{dT}{dt} dS \; dp \quad (14.5.26)$$

Substituting from (14.5.25) in (14.5.26) we get

$$\frac{dE}{dt} = \frac{1}{g} \int \int \int H \; dS \; dp + \frac{1}{g} \int \int \int \omega dS \; dp \quad (14.5.27)$$
From the first two equations of motion including the hydrostatic equation (see (5.5.3) and (5.5.4)) we get

\[
\frac{dK}{dt} = \int \int \int_\Omega \frac{d}{dt} \rho d^2 s \ dz = \frac{1}{\epsilon} \int \int_\Omega \frac{d}{dt} d_\phi \ dS \ dp = \\
- \frac{1}{\epsilon} \int \int_\Omega \vec{v} \cdot \vec{\nabla} \phi \ dS \ dp + \frac{1}{\epsilon} \int \int_\Omega \vec{a} \cdot \vec{\nabla} \cdot \vec{F} \ dS \ dp \tag{14.5.28}
\]

Introducing for simplicity the approximate lower boundary condition \( \omega = 0 \), \( p = p_s \), i.e. disregarding the mountain effect we get

\[
- \frac{1}{\epsilon} \int \int_\Omega \vec{v} \cdot \vec{\nabla} \phi \ dS \ dp = \frac{1}{\epsilon} \int \int_\Omega \vec{\omega} \cdot \vec{v} \ dS \ dp = \\
- \frac{1}{\epsilon} \int \int_\Omega \vec{\omega} \ dS \ dp \tag{14.5.29}
\]

We may therefore write (14.5.27) and (14.5.28) in the form

\[
\frac{dE}{dt} = G(E) - C(E,C) \tag{14.5.30}
\]

\[
\frac{dK}{dt} = C(E,K) - D(K)
\]

where

\[
G(E) = \frac{1}{\epsilon} \int \int_\Omega \int p \ dV \ dS = \int \int H \ p \ dV
\]

\[
D(K) = \frac{1}{\epsilon} \int \int_\Omega \int \vec{a} \cdot \vec{F} \ dS \ dp = - \int \int \vec{v} \cdot \vec{F} \ dV \tag{14.5.31}
\]

\[
C(E,K) = - \frac{1}{\epsilon} \int \int_\Omega \int a \omega \ dS \ dp = \int \int \vec{p} \vec{v} \cdot \vec{\omega} \ dV
\]

where the first expressions should be used in the \( p \)-system and the last in the \( z \)-system.
We have now derived the necessary energy equation to discuss the energy budget of the atmosphere. In the long-term average we have

\[ \int_H \rho dV = \int_V \rho \mathbf{v} \cdot d\mathbf{V} = - \frac{1}{g} \int_s \rho \omega dpds = - \int_V \mathbf{v} \cdot \mathbf{F} dV \quad (14.5.32) \]

Each of these integrals can be used to measure the intensity of the general circulation of the atmosphere. Numerous studies have been made in order to attempt an evaluation of one or several of these integrals, but it is obvious from (14.5.32) that this is a very difficult task. As mentioned before, we get a contribution from the frictional part of \( H \) only in the first integral. An evaluation of the first or the last integral is therefore dependent upon a knowledge of the frictional processes in the atmosphere. The very first attempt to estimate the value of the last integral was made by Brunt who obtained a value of \( 5 \text{ W m}^{-2} \), where we give the dissipation measure per unit area of the earth surface. It is of interest to look somewhat closer at this estimate. We have

\[ D(K) = - \frac{1}{S} \int_s \int_0^\infty \mathbf{v} \cdot \mathbf{F} dz \, ds = - \frac{1}{S} \int_s \int_0^\infty \mathbf{v} \cdot \mathbf{F} \frac{\partial \mathbf{F}}{\partial z} dz \, ds \quad (14.5.33) \]

where \( \mathbf{F} \) is the stress vector. We may now integrate by parts in the last term, and we obtain

\[ D(K) = \frac{1}{S} \int_s \mathbf{v} \cdot \mathbf{F} \, ds + \frac{1}{S} \int_s \int_0^\infty \mathbf{v} \cdot \frac{\partial \mathbf{F}}{\partial z} dz \, ds \quad (14.5.34) \]

In the last integral we may introduce \( \mathbf{v} = \rho \frac{\partial \mathbf{v}}{\partial z} \), and we get

\[ D(K) = \frac{1}{S} \int_s \mathbf{v} \cdot \mathbf{F} \, ds + \frac{1}{S} \int_s \int_0^\infty \mathbf{v} \left( \frac{\partial \mathbf{v}}{\partial z} \right)^2 dz \, ds \quad (14.5.35) \]

Using (14.2.1) and \( \mathbf{v} = c_d \rho_s v_s \), we may write

\[ D(K) = \left[ c_d \rho_s v_s \right] s + \left[ \frac{p_s}{\rho_s} \frac{\partial v_s}{\partial z} \right] s \quad (14.5.36) \]

The first term in (14.5.36) has been evaluated from data to be about \( 2 \text{ W m}^{-2} \) which then is the contribution from the atmospheric boundary layer to the dissipation. The latter term is much more uncertain, but the order of magnitude may be obtained by using \( v = 30 \text{ m s}^{-1} \) and \( 10 \frac{\partial v}{\partial z} \approx 3 \text{ m s}^{-1} \text{ km}^{-1} = 3 \times 10^{-3} \text{ s}^{-1} \).

With these values we get \( 2.7 \text{ W m}^{-2} \) for the last term, which, if correct, would give a total dissipation of \( 4.7 \text{ W m}^{-2} \).
Attempts to evaluate the middle terms in (14.5.32) are hampered by the difficulty in evaluating the horizontal divergence and/or the vertical velocity from atmospheric data. However, the average value for $C(E,K)$ for these methods comes out to be about 2.3 $\text{W m}^{-2}$, or somewhat smaller than the value based upon the frictional dissipation. It must, however, be stressed that a considerable amount of uncertainty exists in all of these estimates. We must therefore be satisfied by saying that the intensity of the general circulation is 2-5 $\text{W m}^{-2}$.

It must also be pointed out that the values quoted above are average values for the major part of the northern hemisphere and for considerable time periods. If one considers individual atmospheric systems over short periods of time (a few days), one finds energy conversion rates which may be several times larger than the values quoted above.

We shall finally make a comparison between the intensity of the general circulation (2-5 $\text{W m}^{-2}$) and the total amount of energy entering the atmosphere. The latter quantity is measured by the solar constant which is the amount of energy received per unit area and per unit time at the outer limit of the atmosphere. It is normally given as:

$$S_o = 1.94 \text{cal cm}^{-2} \text{min}^{-1} = 1354 \text{W m}^{-2}$$

The total amount intercepted by the atmosphere is $S_o a^2$ where $a$ is the radius of the earth. Since the total area of the earth is $4\pi a^2$ we find that the energy received from the sun at the outer limit of the atmosphere and measured per unit area of the surface of the earth is $(1/4)S_o = 338.5 \text{W m}^{-2}$. A good part of this energy is reflected back to space without ever entering the atmosphere. Taking the atmospheric albedo to be 32 per cent there will be 68 per cent which enters the atmosphere or 230 $\text{W m}^{-2}$. Considering the energy conversion rates of the dissipation rates calculated above we find that the efficiency of the atmosphere as an engine is 1-2 per cent.

It is also of importance to have an appreciation of the amounts of energy which we have in the atmosphere. Based upon a study by Oort (1971) of 5 years of data, we may quote the following annual mean values for the northern hemisphere and for the layer from 1000 mb to 75 mb:
Potential energy : \( 567.5 \times 10^6 \text{ J m}^{-2} \)

Internal energy : \( 1674.8 \times 10^6 \text{ J m}^{-2} \)

Total potential energy : \( 2242.3 \times 10^6 \text{ J m}^{-2} \)

Kinetic energy : \( 1153.4 \times 10^3 \text{ J m}^{-2} \)

It is thus seen that the average amount of kinetic energy is about \( 1/2000 \) of the total potential energy \((\text{E} = \text{I} + \text{P})\). We may easily remember the order of magnitude of these quantities by the following estimates:

\[
\text{E} = \frac{c}{gS} \int \int \frac{p}{s} \text{dT dp dS} = \frac{c}{g} \frac{p}{s} \text{T}_M \approx 2500 \times 10^6 \text{ kJ m}^{-2}
\]

where \( \text{T}_M = 250^\circ \text{K} \) is an overall mean temperature for the atmosphere. Similarly

\[
\text{K} = \frac{1}{gS} \int \int \frac{1}{2} v^2 \text{dp dS} = \frac{p}{2g} \left[ \text{v}^2 \right]_M \approx 1125 \times 10^3 \text{ J m}^{-2}
\]

if we select \( v = 15 \text{ m s}^{-1} \) as a reasonable mean value of the wind speed.

The fact that the kinetic energy in the atmosphere is such a small fraction of the total potential energy in the atmosphere is an indication that perhaps not all of the total potential energy can be converted into kinetic energy. This suspicion led Lorenz (1955) to investigate the question, and it was possible for him to show that the total potential energy in the atmosphere has a minimum value attained when the pressure is a constant on each isentropic surface. It is only the total potential energy above the minimum value which can be transformed into kinetic energy. It is beyond the scope of this compendium to discuss the theoretical aspects of these concepts, but it should be mentioned that the amount of total potential energy in the atmosphere minus the minimum value is called the available potential energy, i.e. \( A = E - E_{\text{min}} \) because it is only this rather small part of the total potential energy which can be converted to kinetic energy. It turns out that the available potential energy is only \( 3-4 \) times larger than the kinetic energy.

In addition to the actual intensity of the general circulation of the atmosphere, it is naturally also pertinent to ask how the generations, conversions and dissipations take place. This question has occupied meteorologists since Hadley first tried to explain the trade winds. A considerable amount of research has gone into providing some of the answers to these questions. We shall only mention here that the driving mechanism for the earth's atmosphere seems to be the differential heating between pole and equator. It is in this way that the available potential
energy is generated. For the remaining links in the energetics of the atmosphere it turns out that atmospheric waves, on the various scales, play a most important role in the description of the energy of the waves and of the zonal currents. This viewpoint holds at least for the middle and high latitudes, while the eddies are of minor importance in the tropics where the Hadley cell plays a dominant role for the energy budget of that region.

14.6 The Heat Budget

It is seen from section 14.5 that it is most important to analyse the heat budget of the atmosphere partly for its own sake and partly because of the important role it plays in the energetics of the atmosphere. The ideal situation would be to determine the amount of heating per unit mass and unit time everywhere in the atmosphere, i.e. to determine $H = H(t, \lambda, \varphi, p)$ as $H$ appears in the thermodynamic energy equation. The calculation of the heating function $H$ is a most difficult task because it must be calculated from available atmospheric data and because several very complicated physical processes are at play at the same time and interact with each other. The most important ingredients which together determine the heating of the atmosphere are:

(a) Processes of radiation.
(b) Interaction between the earth and the atmosphere.
(c) Condensation and evaporation processes.

It is necessary to pay attention to all these processes if we are to understand the details of the heat budget for an atmospheric region. However, the heat budget of a given region is not determined by these processes alone. It is also necessary to include the various transport processes by which energy is moved from one place to another. The heat budget is thus also dependent upon the motion of the atmosphere.

In the following we shall for simplicity and clarity restrict ourselves to the heat budget of the zonal rings considered in the preceding section. The basic equation which is used in the analysis is (14.4.19). It links together the heating $H_{NP}$, the storage, and the transport processes. While the storage is of importance when we consider small time periods, it vanishes when we restrict our attention to the annual average, which we shall do in this section. It is then seen that the net heating is equal to the difference in the transports between the northern and southern boundary of the region as expressed in (14.4.25). A detailed knowledge of the transports of sensible heat, potential, and kinetic energy can thus give very important
information on the total non-frictional heating in the zonal ring but not about the contribution from each of the three processes mentioned above. These must be calculated separately, as will be discussed below.

We shall now formulate the heating for the atmosphere \( H_A \). Let \( R_A \) be the excess of absorbed over emitted radiation per unit time and unit mass of the earth's surface. Also let \( Q_s \) be the transfer of sensible heat from the earth to the atmosphere per unit time and unit mass. If \( P \) is the mass of water in precipitation then LP is the transfer of latent heat from the moisture to the atmosphere due to condensation, where \( L \) is the heat of vaporization (\( L = 600 \text{ cal} \, \text{deg}^{-1} \approx 2.5 \times 10^6 \text{ J kg}^{-1} \, \text{deg}^{-1} \)). The total heating of the atmosphere is then

\[
H_A = R_A + Q_s + \frac{E}{p_s} \text{ LP} \quad (14.6.1)
\]

A similar equation can be formulated for the earth surface. If \( R_E \) is the excess radiation, \( E \) the mass of evaporation, LE the transfer of latent heat, then

\[
H_E = R_E - Q_s - \frac{E}{p_s} \text{ LE} \quad (14.6.2)
\]

The terms \( R_A \) and \( R_E \) are obtained from studies of the radiation in the atmosphere (see Physical Meteorology), while \( Q_s \) can be computed from climatological data. The precipitation amounts are obtained from climatological records, while the evaporation can be computed from the treatment of flux of water vapour in the atmospheric boundary layer. It is thus possible to estimate all terms in (14.6.1) and (14.6.2) from theory and data. Without going into details concerning the various ways of computing the terms we shall here consider some of the results. Figure 14.1 shows curves of \( R_A \), \( Q_s \) and LP separately as functions of latitude. It is seen that \( R_A \) is negative from pole to pole without any marked latitudinal variation. The magnitude of \( R_A \) is about \(-1 \times 10^{-2} \text{ J kg}^{-1} \, \text{a}^{-1}\) and creates cooling everywhere.
It is convenient to consider the intensity of a heating or a cooling process by calculating the temperature change which would take place if the process worked all by itself. From the thermodynamic equation we have

\[ \frac{dT}{dt} \sim \frac{H}{c_p} \]  

(14.6.3)

and we find that $1 \times 10^{-2} \text{ J kg}^{-1} \text{ s}^{-1}$ corresponds to approximately 1 deg day$^{-1}$, assuming that one day contains approximately $10^7$ seconds. We may thus say that the cooling rate of $R_A$ is about 1 deg day$^{-1}$. $Q_S$ is positive everywhere except in the high latitudes in both hemispheres, but the absolute value is only 0.1 to 0.3 deg day$^{-1}$. On the other hand, the precipitation is positive everywhere, with the primary maximum in the tropics and secondary maxima in the middle latitudes in both hemispheres. The net atmospheric heating $H_A$, also appearing in Figure 14.13, is calculated from (14.6.1). It shows positive values in the tropics, large negative values in the high latitudes, but also weak negative values in the region around 30° in both hemispheres. In view of (14.6.1) we may say that the cooling in the subtropical regions is related to the relative minima in precipitation in the subtropical anticyclones.

Figure 14.14 shows in a similar way, the components of the heat budget at the surface of the earth. The radiation excess $R_E$ is positive at all latitudes except very close to the poles, and it has a maximum in the equatorial region. $R_E$ is, to a very large extent, balanced by the heating due to evaporation, which is a sink as far as the surface of the earth is concerned, and the transport of sensible heat. The net heating $H_E$ is therefore relatively small (less than 0.05 deg day$^{-1}$), but still with a positive maximum at the equator and negative values in the higher latitudes in both hemispheres.

It follows from Figure 14.13 and Figure 14.14 that a poleward flux must exist in both hemispheres for both the atmosphere and the ocean in order to maintain a long-term balance. The curves for $H_A$ and $H_E$ can, as a matter of fact, be used to calculate the required transport, according to equation (14.4.25), where we may neglect the storage term and put $\phi_2 = \pi/2$. We have, denoting the total flux by $F_A$ for the atmosphere

\[ F_A = -\int_S \int_h H_N P dz \, dS \]  

(14.6.4)
Figure 14.14

$10^{-2} \text{J kg}^{-1} \text{s}^{-1}$

$90^\circ \text{N}$

$R_E$

$H_E$

$Q_S$

$LE$

65-55 45 35 25 15 5 35 45 55 65
We have only one value of $H$ available to us in each zonal ring. It is therefore necessary to consider such a value as a mean value in both the zonal and the vertical directions. Introducing these assumptions in (14.6.4) we find

$$P_A = -\frac{p_s}{g} 2\pi a^2 \int_0^{\pi/2} \left( \frac{H_{NF}}{H_{M}} \right) \cos\phi d\psi$$

(14.6.5)

where the subscripts $Z$ and $M$ refer to the zonal and vertical mean values, respectively. $H_{NF} = H_A$ in (14.6.4) and (14.6.5). The flux $F$ is therefore the combined flux of sensible heat, potential, and kinetic energy, i.e., \( c_p T + \phi + k \), required to obtain a heat balance. We may in a similar way calculate the required energy transport in the earth, i.e., in the oceans, in order to balance $H_E$. This flux $F_0$ must be a heat flux. The two fluxes $F_A$ and $F_0$ are shown in Figure 14.15. However, before we discuss these curves we shall introduce yet another flux, i.e., the flux of latent heat. Let us again consider a volume going from the surface of the earth to the top of the atmosphere. The quantity $E-P$ represents the mass of water which enters the infinitesimal volume. It must therefore be replaced by a flux divergence of the transport of water vapour. We have therefore

$$\rho \frac{E - P}{p_s} L_{E-P} = \nabla \cdot \mathbf{v}_L$$

(14.6.6)

where

$$\mathbf{v}_L = Lq \rho v$$

(14.6.7)

In these expressions we have to use the following interpretations. $(E-P)$ is the net mass entering the atmospheric column per unit area and unit time. $L(E-P)$ is therefore the energy added per unit area and unit time, while the factor $p_s/g$ is the total mass of the atmospheric column such that $(g/p_s)L(E-P)$ becomes the energy added per unit time and unit mass of the atmospheric column. On the other hand, $\nabla \cdot \mathbf{v}_L$ is the transport vector for latent heat, because $q$ is the specific humidity and $L$ the heat of vaporization. (14.6.5) is therefore completely equivalent to (14.4.19), disregarding the storage. It may therefore be integrated to give a meridional flux of latent heat given by

$$P_L = -2\pi a^2 \frac{p_s}{g} \int_0^{\pi/2} \frac{E - P}{p_s} L(E-P) \cos\phi d\psi$$

(14.6.8)
For the flux of sensible heat, potential and kinetic energy we have from (14.6.4)

\[ F_A = -2\pi a^2 \frac{P_S}{\varepsilon} \int_{\varphi}^{\pi/2} \left( R_A + Q_s + \frac{E}{P_S} \right) \cos \varphi d\varphi \]  

(14.6.9)

For the flux in the earth by the oceans we have

\[ F_E = -2\pi a^2 \frac{P_S}{\varepsilon} \int_{\varphi}^{\pi/2} \left( R_E - Q_s - \frac{E}{P_S} \right) \cos \varphi d\varphi \]  

(14.6.10)

Each of the three fluxes \( F_L, F_A \) and \( F_E \) can be computed from the information contained in Figure 14.13 and Figure 14.14. We note in particular that

\[ F_L + F_A + F_E = -2\pi a^2 \frac{P_S}{\varepsilon} \int_{\varphi}^{\pi/2} \left( R_A + R_E \right) \cos \varphi d\varphi \]  

(14.6.11)

This equation says that the total flux in the atmosphere-earth system consisting of the fluxes of latent heat, sensible heat, potential energy, kinetic energy, and the heat flux in the ocean, can be computed from \( (R_A + R_E) \), i.e. from the excess radiation for the atmosphere-earth system.

Figure 14.15 shows the total flux \( (F_L + F_A + F_E) \) and its three components for the whole globe, computed from (14.6.6), (14.6.9), and (14.6.10) and based on Figure 14.13 and Figure 14.14. It is seen that the total flux of energy is poleward in each hemisphere. The same is true for the flux in the oceans \( (F_P) \). The flux of sensible heat, potential, and kinetic energy \( (F_A) \) shows double maxima in each hemisphere. The flux of latent heat has the most complicated appearance because the flux \( (F_L) \) is poleward in middle and high latitudes but equatorward in the low latitudes. The convergence around the equator is due to the fact that \( P \) is larger than \( E \) in the very low latitudes, while the divergence in each hemisphere in the subtropical latitudes comes from the fact that \( E \) is larger than \( P \) in that region.

The information which has gone into the last three figures comes from studies of the radiation in the atmosphere on a global scale, from the calculation of transfer of sensible heat and evaporation from the earth to the atmosphere and from the observed distribution of the precipitation. It should be realized that a considerable amount of uncertainty exists in such calculations based on empirical formulae and scanty observations, especially over the oceans. However, it is believed that these figures give a picture of the heating, its components, and the required fluxes, which is approximately correct in its major features.
This statement can be checked by comparing the results given above with direct calculations of the fluxes from atmospheric data. The fluxes of sensible heat, potential energy, kinetic energy and latent heat have recently been computed by Oort. Figure 14.16 shows a comparison between the two sets of results for the total atmospheric flux $F_A + F_L$, and for each of the components $F_A$ and $F_L$, separately. The comparison can be made for the northern hemisphere only. The curves show that there is good agreement in the general shape of all three curves. The discrepancy is found mainly in the transports going into $F_A$. It should be mentioned that the transport of geopotential energy entering $F_A$ is especially difficult to compute because it depends entirely on the ageostrophic winds, as can be seen from the expression $(\phi v)_z$. The results of Oort show that the flux of potential energy is by no means negligible compared with the flux of sensible heat, and that the flux of $\phi$ in many places is of the opposite sign as compared to the flux of $c_T$. We must therefore conclude that the above fact introduces considerable uncertainty into the calculations based on observations. Oort's calculations show also that the flux of kinetic energy is small compared with the sum of the fluxes of $c_T$ and $\phi$.

We mention finally that observational study can distinguish between the fluxes by the mean meridional calculation and the eddies and that the results confirmed our earlier statements that eddies play a more important role in the middle and higher latitudes than they do in the tropics.

14.7 Mountains, Heating and Friction

The dynamical influence of mountains and heating and the modifying effect of friction are very important on a relatively small time scale (1-2 days), such as in numerical weather prediction, and also on a very long time scale, such as in explaining the quasi-stationary disturbances observed in the atmosphere. We shall here be concerned with both of these aspects.

In order to appreciate the effects of the three physical processes under investigation on the short time scale we shall again turn to the two-level model investigated earlier. The mountain effect enters into this (or any other) model through the lower boundary condition which can be expressed (see section 3.2).

\[ \omega_m = v^* \cdot \nabla h \]  \hspace{1cm} (14.7.1)

where $h = h(x,y)$ is the height of the topography, and $\omega_m$ denotes the forced vertical motion due to the mountains. Since we are going to use pressure as the vertical co-ordinate we shall use

\[ \omega_m \approx - g \rho \hat{w}_m \]  \hspace{1cm} (14.7.2)
Figure 14.16
The effect of friction comes into the model through the specification of the stress. We shall here for simplicity assume that the stress is of importance mainly in the planetary boundary layer, but it is reduced effectively to zero at the top of the planetary boundary layer, i.e. at approximately 1 km. We shall furthermore assume that the stress at the surface is

\[ \vec{t}_s = -c_d \rho_s v_s \vec{v} \]  \hspace{1cm} (14.7.3)

The vorticity equations for the two-level model may now be written:

\[ \frac{\partial \zeta_1}{\partial t} + \vec{v}_1 \cdot \nabla (\zeta_1 + f) = \frac{f_0}{P} \omega_2 \]  \hspace{1cm} (14.7.4)

\[ \frac{\partial \zeta_3}{\partial t} + \vec{v}_3 \cdot \nabla (\zeta_3 + f) = -\frac{f_0}{P} \omega_2 + \frac{f_0}{P} \omega_4 + \frac{\rho}{P} k \cdot \nabla \times \vec{v}_4 \]

In these equations we identify \( \omega_m \) in (14.7.2) with \( \omega_4 \) and \( \vec{t}_s \) with \( \vec{t}_4 \). The direct influence of the mountains is to change the vorticity of the flow at the lower level. If we want to estimate the effect we may measure the change of vorticity created by the mountains in a given time period, say 1 day. We then have

\[ \left( \frac{\partial \zeta}{\partial t} \right)_m = -2 \frac{f_0}{P} \rho_4 \vec{v}_4 \cdot \nabla h = -2 \frac{\sigma f_0}{P T_4} v_{4n} \left| \nabla h \right| \]  \hspace{1cm} (14.7.5)

where \( v_{4n} \) is the velocity component normal to the contours \( h = \text{const} \) (see Figure 14.17). It is thus seen that anticyclonic vorticity will be generated on the windward side of the mountains, while a production of cyclonic vorticity takes place on the lee side. Assuming \( f_0 = 10^{-4} \, \text{s}^{-1} \), \( T_4 = 280^\circ \text{K} \), \( v_{4n} = 10 \, \text{m} \, \text{s}^{-1} \) and \( \left| \nabla h \right| = \frac{1}{2} \times 10^5 \) corresponding to a rise of the terrain by 1 km in 7000 km which is rather steep, we find \( (\Delta \zeta)_m \approx 10^{-4} \, \text{s}^{-1} \) in a 24-hour period. We note that such a slope gives a vertical velocity of \( v_m = 10 \times \frac{1}{2} \times 10^{-2} \, \text{m} \, \text{s}^{-1} = 5 \, \text{cm} \, \text{s}^{-1} \). The effect is therefore very important in numerical forecasting.

![h=const.](image)

\[ \text{Figure 14.17} \]
The magnitude of the frictional effect may be estimated in a similar way. We find from the second equation in (14.7.4) that

\[
\left( \frac{\partial \xi_3}{\partial t} \right)_p = \frac{F}{p} \kappa \cdot \nabla \times \nabla V_4 \approx - \frac{F}{p_4} \frac{c_d P_4 V_4}{R T_4} \xi_3
\]  
(14.7.6)

where we have assumed \( \xi_4 = \frac{1}{2} \xi_3 \) for simplicity. We may write

\[
\left( \frac{\partial \xi_3}{\partial t} \right)_p = - \varepsilon \xi_3
\]  
(14.7.7)

with \( \varepsilon = (\frac{\varepsilon c_d V_4}{RT_4}) \approx 3.6 \times 10^{-6} \text{ s}^{-1} \). The solution of (14.7.7) is

\[
\xi_3 = \xi_{30} e^{-\varepsilon t}
\]  
(14.7.8)

where \( \xi_{30} \) is the initial value of the vorticity. From (14.7.8) we may compute the time it will take for the process of friction to reduce the initial vorticity by a factor of \( e = 2.7 \). We find

\[
T_e = \frac{1}{\varepsilon} = 3.2 \text{ days}
\]  
(14.7.9)

which indicates the order of magnitude of the process of friction.

The heating will naturally influence the temperature field directly. If we wish to consider the effect of the heating on the field of motion, we must incorporate the heating into the vorticity equations (14.7.4). This can be done through the thermodynamic equation applied at level \( 2 \). We get

\[
- \frac{2}{\rho} \frac{\partial \psi_T}{\partial t} - \frac{2}{\rho} \nabla \cdot \nabla \psi_T + \frac{\sigma_2}{\rho_0} \omega_2 = - \frac{R}{c_p} \frac{1}{\rho_0} \rho \frac{2}{\rho_0} H_2
\]  
(14.7.10)

giving

\[
\frac{\sigma_2}{\rho_0} \omega_2 = \beta^2 \left( \frac{\partial \psi_T}{\partial t} + \nabla \cdot \nabla \psi_T \right) - \frac{1}{2} \frac{R}{c_p} \frac{\sigma_2}{\rho_0} H_2
\]  
(14.7.11)

When (14.7.11) is introduced in (14.7.4) we get

\[
\frac{\partial \xi_1}{\partial t} - \beta^2 \frac{\partial \psi_T}{\partial t} + \nabla \cdot \nabla (\xi_1 + \beta - \frac{\sigma_2}{\rho_0} \psi_T) = - \frac{1}{2} \frac{R}{c_p} \frac{\sigma_2}{\rho_0} H_2
\]  
(14.7.12)
\[
\frac{\partial \zeta_3}{\partial t} + \nabla \cdot \vec{v} \left( \zeta_3 + \mathfrak{f} + q^2 \omega_4 \right) = \frac{1}{2} \frac{R}{c_p} \frac{\alpha}{f_0} H_2 + \frac{f}{f_0} \omega_4 - \epsilon \zeta_3 \quad (14.7.13)
\]

The vorticity tendency created by the heating considered in isolation is therefore

\[
\left( \frac{\partial \zeta_1}{\partial t} \right)_H = - \frac{1}{2} \frac{R}{c_p} \frac{\alpha}{f_0} H_2 \quad (14.7.14)
\]

\[
\left( \frac{\partial \zeta_2}{\partial t} \right)_H = + \frac{1}{2} \frac{R}{c_p} \frac{\alpha}{f_0} H_2
\]

We notice that heating will generate cyclonic vorticity at the lower level and anticyclonic vorticity at the higher level, while cooling will have the opposite effects. Using again \( f_0 = 10^{-4} \) \( \text{s}^{-1} \) and \( q^2 = 4 \times 10^{-12} \text{ m}^2 \) we find

\[
\left( \Delta \zeta_3 \right)_H = 4.8 \times 10^{-4} \times H \quad (14.7.15)
\]

where \( \left( \Delta \zeta_3 \right)_H \) is the vorticity change which would be generated in 24 hours if heating were the only process at work. The climatological value of \( H \) is of the order of magnitude: \( 10^{-2} \text{ J kg}^{-1} \text{ s}^{-1} \) giving \( \left( \Delta \zeta_3 \right)_H \approx 0.05 \times 10^{-4} \text{ s}^{-1} \) or, in other words, a very minor change. The heating will therefore be important in cases where the values are much higher than the climatological mean values. The transfer of heat from the surface of the ocean and the heat of condensation can indeed provide a heating rate which is of such a magnitude that it is of importance for a 1-2 day forecast.

Considerations of the type just presented can give a preliminary idea of the rate at which the various processes are working. It must naturally be kept in mind that it is impossible to isolate the processes in reality because of the interplay which always exists between the various processes in the atmosphere. Order of magnitude estimates as presented here are nevertheless necessary to show the importance of each of the factors.

We may get an understanding of the effect of a mountain range by considering as a simple example, a homogeneous atmosphere. The vorticity equation for such an atmosphere is

\[
\frac{d(\zeta + \mathfrak{f})}{dt} = - (\zeta + \mathfrak{f}) \nabla \cdot \vec{v} \quad (14.7.16)
\]
The continuity equation is

\[ \frac{\partial \vec{v}}{\partial z} + \nabla \cdot \vec{v} = 0 \]  \hspace{1cm} (14.7.17)

We shall consider the case of vanishing vertical shear of the horizontal wind, i.e. \( \frac{\partial \vec{v}}{\partial z} = 0 \). In that case \( \nabla \cdot \vec{v} \) is independent of height, and if the depth of the atmosphere is \( H \) we find by integration of (14.7.17) that

\[ \frac{dH}{dt} + H \nabla \cdot \vec{v} = 0 \]  \hspace{1cm} (14.7.18)

Eliminating \( \nabla \cdot \vec{v} \) between (14.7.16) and (14.7.18) we find that

\[ \frac{d}{dt} \left( \frac{\xi + f}{H} \right) = 0 \]  \hspace{1cm} (14.7.19)

which says that the potential vorticity for the homogeneous atmosphere is conserved.

Let us now consider a motion as shown in Figure 14.18. The homogeneous atmosphere is moving from west to east. Far upstream we assume that the depth is \( H_o \) and that the motion is of uniform velocity \( U_o \), which means that \( \xi_o = 0 \). When an atmospheric column comes to the mountain of height \( h \) it will experience a decrease in the depth. According to (14.7.14) it must then decrease its vorticity. We may calculate such changes by using (14.7.19). We obtain:

\[ \frac{\xi + f}{H_o - h} = \frac{\xi_o}{H_o} \]  \hspace{1cm} (14.7.20)

where \( \xi_o \) is the Coriolis parameter at the initial position of the column. Using a beta-plane approximation we find with \( f = f_o + \beta y \) that

\[ \xi = -\beta y - \frac{h}{H_o} f_o \]  \hspace{1cm} (14.7.21)

where \( y = 0 \) at the initial position.
When the column comes to the mountain it will start to gain anticyclonic vorticity \((h > 0)\). If the mountain extends quite far in the southward direction, we can expect that the anticyclonic vorticity will show up as curvature and that the flow will turn to the south. As soon as this happens the column will start to feel the effect of the first term because \(y < 0\). The beta-effect will start to counteract the mountain effect. The farther south the column comes the stronger the beta-effect will be, and it will eventually turn the column back towards the north. It is therefore understandable that a whole train of oscillations in the trajectory of the column can be generated by the mountain, because the beta-effect will continue to act even after the column has left the mountain. We would therefore expect that a ridge (anticyclonic vorticity) would exist on the windward side of the mountain, while a trough (cycloic vorticity) would be created somewhat downstream from the mountain. Explanations of this nature have been offered for the existence of a permanent trough around the east coast of North America and in the region of Japan. The first trough would be generated by the Rocky Mountains, while the second would be the effect of the Himalaya Mountains of Asia. A quantitative evaluation of the mountain effect in the steady state considering the actual topography of the mountains shows that the predicted wave patterns are in reasonable agreement with the flow as seen on the normal maps at 500 mb.

In addition to the mountain effect one may also argue that the effects of the permanent heat sources and sinks in the atmosphere may be responsible for, or at least contribute to, the existence of permanent waves in the atmosphere. It is known from diagnostic calculations of atmospheric heat sources that the western parts of the world oceans are heat sources during winter because of the transfer of sensible heat from the ocean to the cold air masses which come from the continent in the westerlies. The effect of the heat sources would be to generate cycloic vorticity
over the heat sources, while anticyclonic vorticity would be produced over the continents during winter because the continents are the heat sinks during the cold season. It should be mentioned that the heat source effect will change sign during the seasons, while the mountain effect will change in intensity only because of the somewhat weaker winds during summer.

One may investigate the importance of the factors mentioned above using the equations for the two-level quasi-geostrophic model in the steady state case. The pertinent equations are (14.7.12) and (14.7.13) in which we set the local time derivatives equal to zero. It is convenient to add and subtract the two equations. After division by 2 we get:

$$\nabla T \cdot \nabla \zeta_T = - \frac{f_0}{p_4} \varepsilon T \cdot \nabla h - \frac{1}{2} \varepsilon T$$  \hspace{1cm} (14.7.22)

$$- q^2 \nabla \zeta_T \cdot \nabla h + \frac{1}{2} \varepsilon T$$  \hspace{1cm} (14.7.23)

where the subscripts $\ast$ and $T$ have the same meanings as in section 11.9. These equations are most easily solved if we linearize around a state characterized by $U_\ast = \text{const.}$ and $U_T = \text{const.}$ The equations become linear equations which can be solved by standard methods. Assuming $\nabla T = \nabla T - 2 \nabla T$, $\zeta_T - \zeta_\ast = \zeta_T$ we find

$$\left( \frac{\partial V^2}{\partial x} + \frac{c}{\varepsilon T} \frac{\partial V}{\partial x} \right) + \left( \frac{\partial V^2}{\partial x} - \frac{1}{2} \beta \frac{\partial V}{\partial x} \right) =$$

$$- \frac{c}{\varepsilon T} \left( U_\ast - 2 U_T \right) \frac{\partial h}{\partial x}$$  \hspace{1cm} (14.7.24)

$$\left( \frac{\partial V^2}{\partial x} - \frac{c}{\varepsilon T} \frac{\partial V}{\partial x} \right) + \left( \frac{\partial V^2}{\partial x} + \left( \beta - \frac{c}{\varepsilon T} \right) \frac{\partial V}{\partial x} + \frac{c}{\varepsilon T} \right)$$

$$= \frac{c}{\varepsilon T} \left( U_\ast - 2 U_T \right) \frac{\partial h}{\partial x} - \frac{1}{2} \frac{\partial V}{\partial x} \frac{c}{\varepsilon T}$$  \hspace{1cm} (14.7.25)

These equations are solved using the assumption that

$$\left[ \psi, \psi_T, H, h \right] = \left[ \psi_\ast(x), \psi_T(x), H(x), h(x) \right] \cos \mu y$$  \hspace{1cm} (14.7.26)

where $\mu$ is a measure of the scale in the south-north direction. In the calculations described in the following we take $\mu = 0.95 \times 10^{-6}$ m$^{-1}$ corresponding to a meridional wavelength of 60$^\circ$ of latitude. The actual solution of the coupled equations (14.7.24)
and (14.7.25) is then accomplished by expanding the given functions $H'$ and $h'$ and the unknown functions $\psi_0$ and $\psi_n$ in Fourier series with respect to $x$. We shall not report all the details of the solution but consider some of the results.

We note then first of all that by setting $h'(x) = 0$ we can investigate the effects of the heating, while $H'(x) = 0$ will give us the effect of the mountains by themselves. The general case is of course the case where $h'(x)$ and $H'(x)$ are different from zero. Figure 14.19 shows the heating function from diagnostic calculations as a function of longitude at $45^\circ N$ for the month of January. The positions of the continents are shown in a schematic way. It is seen that the cooling is found mainly over the continents while the heating is over the oceans. The magnitude is of the same order as we found earlier for the averaged heating in the zonal rings.

Figure 14.20 shows the profile of the topography also along $45^\circ N$. The outstanding features here are naturally the major mountain ranges in Asia and North America.

Figure 14.21 shows the predicted profile of the heights of the 250, 500 and 750 mb surfaces for the case of $H(x) = 0$, i.e. no heating effect. The following values of the parameters were used: $U_w = 15 \text{ m s}^{-1}$, $U_p = 5 \text{ m s}^{-1}$, $\varepsilon = 4 \times 10^{-6} \text{ s}^{-1}$, $\nu^2 = 0.95 \times 10^{-12} \text{ m}^{-2}$. An inspection of the figure shows that the largest effect is found at the upper level (250 mb). It is furthermore seen that the trough around Japan is somewhat deeper than the trough along the East coast of North America. Note furthermore that the systems show essentially no slope in the vertical direction.

Figure 14.22 shows the perturbations generated by the heating alone, i.e. $h'(x) = 0$, $H'(x) \neq 0$. We find by a comparison of this figure with the previous figure that the positions of troughs and ridges are approximately the same in the two cases. However, there are other important differences. The first is that the heating results in perturbations of smaller amplitude than those generated by the mountains. The second difference is that the perturbations generated by the heating show a marked slope in the vertical direction from east to west as we go up through the atmosphere. The major result is therefore that the effects of mountains and heating reinforce each other in the calculation for January, and that the qualitative arguments presented earlier for the importance of mountains and the heat sources turn out to be correct.

Figure 14.23 shows the combined effect of both mountains and heating in the two broken-line curves of which the dashed curve has $\varepsilon = 4 \times 10^{-6} \text{ s}^{-1}$, while the dash-dotted curve has $\varepsilon = 6 \times 10^{-6} \text{ s}^{-1}$. The solid curve is the actual distribution of
the contours on the normal maps for the three isobaric levels. On the basis of Figure 14.23 one may conclude that it is possible to describe the major feature of the observed stationary disturbances using the simple two-level model incorporating mountains, heating and friction.

Figure 14.21
Figure 14.22
Figure 14.23
APPENDIX

VECTOR SYMBOLS AND VECTOR ALGEBRA

A.1 Introduction

We experience two kinds of quantities in meteorology, scalars and vectors. A scalar is a quantity characterized by magnitude only. Examples are: pressure, density, temperature, humidity. A vector is a quantity which is characterized by magnitude and direction. Examples are: force, velocity and acceleration. The present compendium will make use of the vector notation (in this case letters with an arrow) such as \( \vec{A} \), and it is necessary to know the rules of vector algebra in order to follow the derivations given in the text. A short summary of vector algebra is given in this appendix.

A.2 Sum, Difference, Multiplication by a Scalar

![Figure A.1](image)

The given vectors are \( \vec{A} \) and \( \vec{B} \). Figure A.1 shows the geometrical construction necessary to form a vector \( \vec{C} \) which is called the sum of \( \vec{A} \) and \( \vec{B} \), i.e.

\[
\vec{C} = \vec{A} + \vec{B}
\]  \hspace{2cm} (A.2.1)

The vector \( -\vec{A} \) is a vector directed opposite to \( \vec{A} \) but with the same length as \( \vec{A} \). The difference \( \vec{D} \) between two vectors, \( \vec{A} \) and \( \vec{B} \), is defined by the relation

\[
\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})
\]  \hspace{2cm} (A.2.2)

as shown in Figure A.2.

![Figure A.2](image)
The symbol $k \vec{A}$ means a vector which is of the length $kA$ and directed in the same direction as $\vec{A}$ if $k > 0$, but opposite to $\vec{A}$ if $k < 0$ (see Figure A.3). This operation is called the multiplication of a vector by a scalar.

![Figure A.3](image)

A unit vector is a vector of unit length. There are infinitely many unit vectors, one for each direction we care to specify. If $\hat{a}$ is a unit vector in the same direction as $\vec{A}$ we may write

$$\vec{A} = A \hat{a}$$  \hspace{1cm} (A.2.3)

because the two vectors $\vec{A}$ and $A \hat{a}$ have the same direction and the same length, i.e. $A$. This way of writing a vector is very useful in many applications which make use of a co-ordinate system. Let $(x, y, z)$ be an ordinary Cartesian co-ordinate system (see Figure A.4), and let $\vec{A}$ be an arbitrary vector. We may first resolve $\vec{A}$ into a horizontal vector $\vec{A}_h$ and a vertical vector $\vec{A}_z$, i.e.

$$\vec{A} = \vec{A}_h + \vec{A}_z$$  \hspace{1cm} (A.2.4)

![Figure A.4](image)
The vector \( \vec{A}_h \) may then be resolved in a vector along the x-axis and one along the y-axis, i.e. \( \vec{A}_x \) and \( \vec{A}_y \), respectively. Thus

\[
\vec{A}_h = \vec{A}_x + \vec{A}_y
\]  

(A.2.5)

and by substitution in (A.2.4)

\[
\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z
\]

(A.2.6)

Let us next introduce three unit vectors: \( \vec{i} \), \( \vec{j} \) and \( \vec{k} \) in the x-, y- and z-directions, respectively. According to (A.2.3) we have

\[
\vec{A}_x = A_x \vec{i}, \vec{A}_y = A_y \vec{j}, \vec{A}_z = A_z \vec{k}
\]

(A.2.7)

and we may write (A.2.6) in the form

\[
\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}
\]

(A.2.8)

If, in addition,

\[
\vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k}
\]

(A.2.9)

we find, using the rules for addition and subtraction

\[
\vec{A} + \vec{B} = (A_x + B_x)\vec{i} + (A_y + B_y)\vec{j} + (A_z + B_z)\vec{k}
\]

(A.2.10)

and

\[
\vec{A} - \vec{B} = (A_x - B_x)\vec{i} + (A_y - B_y)\vec{j} + (A_z - B_z)\vec{k}
\]

(A.2.11)

From geometrical considerations or from (A.2.10) it is easily seen that

\[
\vec{A} + \vec{B} = \vec{B} + \vec{A}
\]

A.3 Multiplication of Vectors

There are two products of two vectors: (a) The scalar product and (b) The vector product.
The scalar product of the two vectors \( \vec{A} \) and \( \vec{B} \) is a scalar of the magnitude \( AB \cos \alpha \) where \( \alpha \) is the angle from \( \vec{A} \) to \( \vec{B} \). We write

\[
C = \vec{A} \cdot \vec{B} = AB \cos \alpha \quad (A.3.1)
\]

Since \( \cos(-\alpha) = \cos \alpha \) it follows that

\[
\vec{B} \cdot \vec{A} = \vec{A} \cdot \vec{B} \quad (A.3.2)
\]

We note that the scalar product of two vectors is zero when the two vectors are perpendicular to each other. The scalar product of a vector by itself is

\[
\vec{A} \cdot \vec{A} = A^2 \quad (A.3.3)
\]

because \( \cos 0 = 1 \).

It follows from these results that

\[
\vec{1} \cdot \vec{1} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \quad (A.3.4)
\]

while

\[
\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0 \quad (A.3.5)
\]

Using these rules and \((A.3.2)\) it follows by direct calculation that

\[
\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad (A.3.6)
\]

when \( A \) and \( B \) are given by \((A.2.3)\) and \((A.2.9)\), respectively.

Using \((A.3.6)\) and \((A.3.10)\) it can be shown that

\[
\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \quad (A.3.7)
\]

The vector product of two vectors \( \vec{A} \) and \( \vec{B} \) is a vector \( \vec{C} \) with the magnitude \( AB \sin \alpha \), where \( \alpha \) is the angle from \( \vec{A} \) to \( \vec{B} \), and the direction such that \( \vec{A} \), \( \vec{B} \) and \( \vec{C} \) (in that order) form a right-handed system. The symbol is

\[
\vec{C} = \vec{A} \times \vec{B} \quad (A.3.8)
\]
Figure A.5 shows the vectors \( \vec{A}, \vec{B} \) and \( \vec{C} = \vec{A} \times \vec{B} \). It is seen that the magnitude of the vector product is equal to the area of the parallelogram with the adjoining sides equal to \( \vec{A} \) and \( \vec{B} \).

It is also seen that
\[
\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}
\]  \hspace{1cm} (A.3.9)

The vector product of a vector by itself is the zero vector because \( \sin 0 = 0 \). It follows that
\[
\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0
\]  \hspace{1cm} (A.3.10)

Using the definition of the vector product and Figure A.6 it is seen that

\[
\vec{i} \times \vec{j} = \vec{k}
\vec{j} \times \vec{k} = \vec{i}
\vec{k} \times \vec{i} = \vec{j}
\]  \hspace{1cm} (A.3.11)

It follows then from (A.3.11) and (A.3.9) that
\[
\vec{j} \times \vec{i} = -\vec{k}, \quad \vec{k} \times \vec{j} = \vec{i}, \quad \vec{i} \times \vec{k} = -\vec{j}
\]  \hspace{1cm} (A.3.12)
The rules given in (A.3.11) and (A.3.12) make it possible to evaluate the vector product in terms of the components of each vector. Thus

\[
\mathbf{a} \times \mathbf{b} = (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \times (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k})
\]

\[
= -A_y B_z \mathbf{i} - A_z B_y \mathbf{j} - A_x B_z \mathbf{k} + A_x B_y \mathbf{i} + A_y B_z \mathbf{j} + A_z B_x \mathbf{k}
\]

\[
= (A_y B_z - A_z B_y) \mathbf{i} - (A_z B_x - A_x B_z) \mathbf{j} + (A_x B_y - A_y B_x) \mathbf{k}
\]

where the symbolic determinant is written in order to form a mnemonic rule.

It can be shown from (A.3.13) that

\[
\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}
\]

(A.3.14)

A.4 Multiplication Involving More than Two Vectors

It is possible to form many products involving three or more vectors using the scalar and the vector products defined in Section A.3. We shall restrict ourselves to those combinations of special importance in meteorology.

Let us first consider

\[
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}
\]

(A.4.1)

It is obvious that the result of this combined operation is a scalar which is formed when we first multiply \(\mathbf{a}\) vectorially by \(\mathbf{b}\). We form then the scalar product of \(\mathbf{c}\) and the resultant vector of the operation \(\mathbf{a} \times \mathbf{b}\).

The product formed in (A.4.1) has a geometrical interpretation which can be obtained from Figure A.7. The magnitude of \(\mathbf{A} \times \mathbf{B}\) is the area of the parallelogram formed by these vectors.
On the other hand, the magnitude of the final vector is

$$| \vec{A} \times \vec{B} | \cdot | \vec{C} | \cdot \cos \theta$$  \hspace{1cm} (A.4.2)

where $| \cdot |$ means the magnitude of the vector. However $| \vec{C} | \cos \theta$ is the distance from the bottom to the top of the box formed by $\vec{A}$, $\vec{B}$ and $\vec{C}$. It follows therefore that the expression in (A.4.2) is the volume of the box in Figure A.7, because $| \vec{A} \times \vec{B} |$ is the area of the bottom surface and $| \vec{C} | \cos \theta$ is the distance.

Using (A.3.13) and (A.3.6) we get

$$\begin{align*}
(\vec{A} \times \vec{B}) \cdot \vec{C} &= \left[ (A_y B_z - A_z B_y) \vec{I} - (A_x B_z - A_z B_x) \vec{J} + (A_x B_y - A_y B_x) \vec{K} \right] \cdot (C_x \vec{I} + C_y \vec{J} + C_z \vec{K}) \\
&= (A_y B_z - A_z B_y) C_x - (A_x B_z - A_z B_x) C_y + (A_x B_y - A_y B_x) C_z \\
&= \begin{vmatrix}
A_x & A_y & A_z \\
B_x & B_y & B_z \\
C_x & C_y & C_z
\end{vmatrix}  \hspace{1cm} (A.4.3)
\end{align*}$$

where the determinant is the easier way to remember the evaluation of the product.

From (A.4.3) or from the geometrical interpretation as a volume we find that

$$\begin{align*}
(\vec{A} \times \vec{B}) \cdot \vec{C} &= (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B} \\
(\vec{A} \times \vec{B}) \cdot \vec{C} &= \vec{A} \cdot (\vec{B} \times \vec{C}) \\
(\vec{A} \times \vec{B}) \cdot \vec{C} &= \vec{A} \cdot (\vec{C} \times \vec{A}) \cdot \vec{B} \\
(\vec{A} \times \vec{B}) \cdot \vec{C} &= \vec{A} \cdot \vec{B} \cdot \vec{C} \\
(\vec{A} \times \vec{B}) \cdot \vec{C} &= \vec{A} \cdot \vec{B} \cdot \vec{C}
\end{align*}$$  \hspace{1cm} (A.4.4)
which says that the vectors may be rotated in a cyclical manner in products of the type (A.4.1). We also get:

\[(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{C} \cdot (\vec{A} \times \vec{B}) = - \vec{C} \cdot (\vec{B} \times \vec{A})\]

(A.4.5)
as an example of the many combinations which are possible.

We shall next consider the product

\[\vec{A} \times (\vec{B} \times \vec{C})\]

(A.4.6)

which is a vector formed by first forming \(\vec{B} = \vec{B} \times \vec{C}\) and then \(\vec{A} \times \vec{B}\). Note here that the parenthesis in (A.4.6) is very important because it tells the order in which the operations should be performed. As we shall presently show, (A.4.6) is not in general the same as \((\vec{A} \times \vec{B}) \times \vec{C}\) which indicates that the parenthesis must be retained. That the two products \(\vec{A} \times (\vec{B} \times \vec{C})\) and \((\vec{A} \times \vec{B}) \times \vec{C}\) are different can be seen by an example. We calculate that

\[\vec{I} \times (\vec{I} \times \vec{J}) = \vec{I} \times \vec{K} = - \vec{J}\]

(A.4.7)

-while \((\vec{I} \times \vec{I}) \times \vec{J}\) is the zero vector and therefore different from \(- \vec{J}\).

It is possible to resolve the vector (A.4.6) into two vectors along \(\vec{B}\) and \(\vec{C}\). We shall prove that

\[\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}\]

(A.4.8)

The proof is straightforward, but somewhat laborious. By calculating first the left hand side and then the other side on component form, we get

\[\vec{B} \times \vec{C} = (B_{x}C_{y} - B_{y}C_{x})\vec{I} - (B_{x}C_{z} - B_{z}C_{x})\vec{J} + (B_{x}C_{y} - B_{y}C_{x})\vec{K}\]

and from the expression for \(\vec{B} \times \vec{C}\) we form

\[\vec{A} \times (\vec{B} \times \vec{C}) = \left[ A_{y}(B_{x}C_{y} - B_{y}C_{x}) + A_{z}(B_{x}C_{z} - B_{z}C_{x}) \right] \vec{I}\]

\[+ \left[ A_{x}(B_{y}C_{z} - B_{z}C_{y}) + A_{z}(B_{y}C_{y} - B_{y}C_{z}) \right] \vec{J}\]

\[- \left[ A_{x}(B_{z}C_{x} - B_{x}C_{z}) + A_{y}(B_{z}C_{y} - B_{y}C_{z}) \right] \vec{K}\]
On the other hand, we have

\[
(\vec{A} \cdot \vec{C}) \vec{B} = (\vec{A} \cdot \vec{B}) \vec{C}
\]

\[
= (A_x C_x + A_y C_y + A_z C_z) \left( B_x \vec{i} + B_y \vec{j} + B_z \vec{k} \right)
\]

\[
- (A_x B_x + A_y B_y + A_z B_z) \left( C_x \vec{i} + C_y \vec{j} + C_z \vec{k} \right)
\]

\[
= \left[ A_y (B_x C_y - B_y C_x) + A_z (B_x C_z - B_z C_x) \right] \vec{i}
\]

\[
+ \left[ A_x (B_y C_x - B_x C_y) + A_z (B_y C_z - B_z C_y) \right] \vec{j}
\]

\[
- \left[ A_x (B_C x - B_x C_x) + A_y (B_C y - B_y C_y) \right] \vec{k}
\]

A comparison between the two final expressions shows that (A.4.8) is true. We can use this formula to calculate

\[
(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{C} \times (\vec{A} \times \vec{B}) = -(\vec{C} \cdot \vec{B}) \vec{A} + (\vec{C} \cdot \vec{A}) \vec{B}
\]  \hspace{1cm} (A.4.9)

which again demonstrates that \((\vec{A} \times \vec{B}) \times \vec{C} \neq \vec{A} \times (\vec{B} \times \vec{C})\).

A.5 The Operator \(\nabla\)

It is convenient to introduce a symbolic vector, \(\nabla\), defined by the expression

\[
\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}
\]  \hspace{1cm} (A.5.1)

The symbolic vector is called by various names such as "del", "nabla", "grad" or the Hamilton operator. "Del" is probably the name used by most people.

If we apply (A.5.1) to an arbitrary scalar \(b\) we get a vector with the components \(\partial b/\partial x\), \(\partial b/\partial y\), \(\partial b/\partial z\), or

\[
\nabla b = \frac{\partial b}{\partial x} \vec{i} + \frac{\partial b}{\partial y} \vec{j} + \frac{\partial b}{\partial z} \vec{k}
\]  \hspace{1cm} (A.5.2)
We obtain an interpretation of \( \nabla b \) if we consider the variation of \( b \) between two arbitrary points: \((x, y, z)\) and \((x + \delta x, y + \delta y, z + \delta z)\). We find

\[
\delta b = b(x + \delta x, y + \delta y, z + \delta z) - b(x, y, z) = b(x, y, z) + \frac{\partial b}{\partial x} \delta x + \frac{\partial b}{\partial y} \delta y + \frac{\partial b}{\partial z} \delta z - b(x, y, z)
\]

\[= \nabla b \cdot \delta \vec{r} \quad (A.5.3)\]

where \( \delta \vec{r} \) is the vector with components \( \delta x, \delta y, \delta z \).

We note two special cases. If the two points are selected in the same equisclar surface of \( b \), i.e. \( \delta b = 0 \), we find

\[\nabla b \cdot \delta \vec{r} = 0 \quad (A.5.4)\]

but \( \delta \vec{r} \) is a vector in the equisclar surface, and since the scalar product is zero, we find the important result that \( \nabla b \) is perpendicular to the equisclar surface (see Figure A.3).

In the next case, Figure A.9, we select the two points along the same normal to the equisclar surface. The first point is in the equisclar surface \( b = \text{const.} \), the second point in the surface \( b + \delta b = \text{const.} \). We have

\[\delta b = \nabla b \cdot \delta \vec{n} \quad (A.5.5)\]

where \( \delta \vec{n} \) is the vector from point 1 to point 2 along \( \nabla b \). The two vectors \( \nabla b \) and \( \delta \vec{n} \), have therefore the same direction, and (A.5.5) may be written

\[|\nabla b| = \frac{\delta b}{\delta n} \quad (A.5.6)\]

which shows how we can evaluate the magnitude of \( \nabla b \).

![Figure A.5](image-url)
The \( \mathbf{\nabla} \)-operator can also be applied to a vector. If

\[
\mathbf{\vec{A}} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}
\]  

we find for the scalar product of \( \mathbf{\nabla} \) and \( \mathbf{\vec{A}} \):

\[
\mathbf{\nabla} \cdot \mathbf{\vec{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}  
\]  

The particular combination appearing in (A.5.8) is called the **divergence** of the vector field \( \mathbf{\vec{A}} \). The kinematic interpretation of (A.5.8) is given in the main text (see Section 6.10). Occasionally, one writes:

\[
\text{div } \mathbf{\vec{A}} = \mathbf{\nabla} \cdot \mathbf{\vec{A}}
\]  

We may also form the vector product of \( \mathbf{\nabla} \) and \( \mathbf{\vec{A}} \) and obtain

\[
\mathbf{\nabla} \times \mathbf{\vec{A}} = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{array} \right|
\]

\[
= \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{i} - \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \mathbf{j} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k}
\]

The vector \( \mathbf{\nabla} \times \mathbf{\vec{A}} \) is sometimes called the **curl** of \( \mathbf{\vec{A}} \) and written

\[
\text{curl } \mathbf{\vec{A}} = \mathbf{\nabla} \times \mathbf{\vec{A}}
\]
If a special case, is the velocity vector \( \vec{A} = \vec{v} \) with the components \( u, v \) and \( w \), we find from (A.5.1) and (A.5.10) that

\[
\text{div } \vec{v} = \nabla \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{A.5.12}
\]

and

\[
\text{curl } \vec{v} = \nabla \times \vec{v} = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k} \tag{A.5.13}
\]

The vector \( \text{curl } \vec{v} = \nabla \times \vec{v} \) is called the vorticity vector and is of special significance in meteorology. We note in particular that the vertical component of \( \nabla \times \vec{v} \) can be written as

\[
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \hat{k} \cdot (\nabla \times \vec{v}) \tag{A.5.14}
\]

The \( \nabla \)-operator can be used in many ways. We may for example evaluate

\[
\nabla \cdot (\vec{B} \times \vec{C}) \quad \text{and we shall prove the formula}
\]

\[
\nabla \cdot (\vec{B} \times \vec{C}) = - \vec{B} \cdot (\nabla \times \vec{C}) + \vec{C} \cdot (\nabla \times \vec{B}) \tag{A.5.15}
\]

The proof is obtained by evaluation, using (A.4.3)

\[
\nabla \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
B_x & B_y & B_z \\
x & y & z \\
C_x & C_y & C_z \\
\end{vmatrix} = \frac{\partial}{\partial x} \left( B_y C_z - B_z C_y \right) - \frac{\partial}{\partial y} \left( B_x C_z - B_z C_x \right) + \frac{\partial}{\partial z} \left( B_x C_y - B_y C_x \right) \tag{A.5.16}
\]

Differentiating out in all the terms we obtain

\[
\nabla \cdot (\vec{B} \times \vec{C}) = B_x \frac{\partial C_z}{\partial x} + C_x \frac{\partial B_z}{\partial x} - B_z \frac{\partial C_x}{\partial x} - C_z \frac{\partial B_x}{\partial x}
\]

\[
- B_y \frac{\partial C_z}{\partial y} - C_y \frac{\partial B_z}{\partial y} + B_z \frac{\partial C_y}{\partial y} + C_z \frac{\partial B_y}{\partial y}
\]

\[
+ B_x \frac{\partial C_y}{\partial z} + C_x \frac{\partial B_y}{\partial z} - B_y \frac{\partial C_x}{\partial z} - C_y \frac{\partial B_x}{\partial z}
\]
\[- B \cdot (\nabla \times \vec{\mathcal{E}}) + \vec{C} \cdot (\nabla \times \vec{B}) \tag{A.5.17} \]

The following operation gives
\[
\nabla \cdot (\nabla b) = \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} + \frac{\partial^2 b}{\partial z^2} = \text{div} (\text{grad} b) \tag{A.5.18} 
\]

It is customary to write
\[
\nabla \cdot (\nabla b) = \nabla^2 b = \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} + \frac{\partial^2 b}{\partial z^2} \tag{A.5.19} \]

and the operator \(\nabla^2\) is called the Laplacian. It has the same name even if we restrict ourselves to two dimensional fields, i.e. \(b = b(x,y)\), in which case
\[
\nabla \cdot (\nabla^2 b) = \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} = \nabla^2 b \tag{A.5.20} 
\]

The Laplacian is the divergence of the del-operator. If we on the other hand take the curl of the del-operator we get
\[
\nabla \times (\nabla b) = 0 \tag{A.5.21} 
\]

because
\[
\nabla \times (\nabla b) = \begin{vmatrix}
\hat{\imath} & \hat{\jmath} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} & \frac{\partial b}{\partial z}
\end{vmatrix} = - \hat{\imath} \left( \frac{\partial^2 b}{\partial y \partial z} - \frac{\partial^2 b}{\partial z \partial y} \right) - \hat{\jmath} \left( \frac{\partial^2 b}{\partial z \partial x} - \frac{\partial^2 b}{\partial x \partial z} \right) + \hat{k} \left( \frac{\partial^2 b}{\partial x \partial y} - \frac{\partial^2 b}{\partial y \partial x} \right) \tag{A.5.22} 
\]

In meteorological equations we have terms of the form \(b \nabla c\). It is therefore of interest to calculate \(\nabla \cdot (b \nabla c)\) and \(\nabla \times (b \nabla c)\). We find for the first of these:
$$\nabla \cdot (b \nabla c) = \frac{\partial}{\partial x} \begin{pmatrix} b & \frac{dc}{dx} \\ \frac{dc}{dx} & \frac{d}{dy} \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} b & \frac{dc}{dy} \\ \frac{dc}{dy} & \frac{d}{dz} \end{pmatrix} + \frac{\partial}{\partial z} \begin{pmatrix} b & \frac{dc}{dz} \\ \frac{dc}{dz} & \frac{d}{dz} \end{pmatrix}$$

$$= b \nabla^2 c + \nabla b \cdot \nabla c \quad (A.5.23)$$

while the second becomes

$$\nabla \times (b \nabla c) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{dc}{dx} & \frac{dc}{dy} & \frac{dc}{dz} \end{vmatrix}$$

$$= \hat{i} \left\{ \frac{\partial}{\partial y} \frac{dc}{dz} - \frac{\partial}{\partial z} \frac{dc}{dy} \right\} - \hat{j} \left\{ \frac{\partial}{\partial x} \frac{dc}{dz} - \frac{\partial}{\partial z} \frac{dc}{dx} \right\} + \hat{k} \left\{ \frac{\partial}{\partial x} \frac{dc}{dy} - \frac{\partial}{\partial y} \frac{dc}{dx} \right\}$$

$$= \nabla b \times \nabla c \quad (A.5.24)$$

or, in summary

$$\nabla \cdot (b \nabla c) = b \nabla^2 c + \nabla b \cdot \nabla c \quad (A.5.25)$$

$$\nabla \times (b \nabla c) = \nabla b \times \nabla c \quad (A.5.26)$$
NOTE

Volume I of the Compendium of Meteorology has been published in two separate parts:

Volume I - PART 1 Dynamic Meteorology by A. Wiin-Nielsen

and

Volume I - PART 2 Physical Meteorology by B.J. Retallack